Effective metrics in the non-minimal Einstein-Yang-Mills-Higgs theory

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We formulate a self-consistent non-minimal five-parameter Einstein-Yang-Mills-Higgs (EYMH) model and analyse it in terms of effective (associated, color and color-acoustic) metrics. We use a formalism of constitutive tensors in order to reformulate master equations for the gauge, scalar and gravitational fields and reconstruct in the algebraic manner the so-called associated metrics for the Yang-Mills field. Using WKB-approximation we find color metrics for the Yang-Mills field and color-acoustic metric for the Higgs field in the framework of five-parameter EYMH model. Based on explicit representation of these effective metrics for the EYMH system with uniaxial symmetry, we consider cosmological applications for Bianchi-I, FLRW and de Sitter models. We focus on the analysis of the obtained expressions for velocities of propagation of longitudinal and transversal color and color-acoustic waves in a (quasi)vacuum interacting with curvature; we show that curvature coupling results in time variations of these velocities. We show, that the effective metrics can be regular or can possess singularities depending on the choice of the parameters of non-minimal coupling in the cosmological models under discussion. We consider a physical interpretation of such singularities in terms of phase velocities of color and color-acoustic waves, using the terms "wave stopping" and "trapped surface".

I. INTRODUCTION

The coupled system of Einstein-Yang-Mills-Higgs (EYMH) equations form a mathematical basis for the well-known self-consistent model of interaction of gravitational, gauge and scalar fields (see, e.g., [1] for review and basic references). The Einstein-Yang-Mills-Higgs theory unifies two important trends in modern theory of gravity. The first one is represented by the so-called Einstein-Yang-Mills model, which can be considered as a non-Abelian generalization of the Einstein-Maxwell model. The second trend is connected with the investigations of interaction of gravitational and scalar fields. $SU(n)$ symmetric EYMH theory synthesizes the ideas and methods elaborated in both models, inherits well-known results and presents some qualitatively new ones. One of the most interesting results obtained in the framework of the Einstein-Yang-Mills-Higgs model is connected with the search for exact solutions describing non-Abelian field configurations of the monopole type (see, e.g., [2]-[9]). Other important applications of the EYMH theory are the modeling of cosmological evolution (e.g., [10, 11]) and modeling of particle dynamics in the field of non-Abelian monopoles (e.g., [12]). Why is the EYMH model interesting for cosmological applications? As it was stressed in [10], numerous models of inflation in the Early Universe involve into consideration the multiplet of Higgs scalar fields $\mathbf{\Phi}$. Higgs scalar fields are coupled with the gauge Yang-Mills field, characterized by potential four-vectors $\mathbf{A}_k$, the so-called minimal coupling being realized by means of the gauge-covariant derivative $\hat{D}_m \mathbf{\Phi} = \partial_m \mathbf{\Phi} + [\mathbf{A}_m, \mathbf{\Phi}]$. Since contributions of the Higgs fields into the total stress-energy tensor of the system are considered in many models as the dominating ones at the inflation stage, the gauge counterpart of the Higgs field, the Yang-Mills field, should also be included into the master equations. As for the present stage of the Universe evolution, the models with Higgs fields appear, first, in the theory of dark matter [13, 14], second, in the theories of dark energy [15, 16]. Just the discovery of accelerated expansion of the present Universe revived an interest in *non-minimal* Einstein-Yang-Mills-Higgs models in the context of search for an explanation of the dark energy phenomenon. One of the most advanced trends of such type is represented by the so-called $f(R)$ theories (e.g., [17]), which are based on the appropriate (nonlinear) modifications of the Einstein-Hilbert part of the total Lagrangian ($R/8\pi G$) linear in the Ricci scalar. Another version of non-minimal extension of the EYMH theory is characterized by the non-minimal modification of a scalar field contribution to the total Lagrangian (e.g., [18]). Our approach to the non-minimal modification of the EYMH theory is presented below.

The discussion concerning the non-minimal coupling of gravity with fields and media started at the end of the

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60th. The historical details, review and references related to the non-minimal interaction of gravity with scalar and electromagnetic fields can be found, e.g., in [19, 20, 21]. As for the generalization of the concept of curvature coupling for the case of gauge field, there are two different ways to establish a non-minimal Einstein-Yang-Mills theory. The first way is a direct non-minimal generalization of the Einstein-Yang-Mills (EYM) theory containing derivatives of the second order at most [22]. In the framework of this approach Müller-Hoissen obtained the non-minimal EYM model from a dimensional reduction of the Gauss-Bonnet action [23]. We follow the alternative way, which is connected with a non-Abelian generalization of the non-minimal Einstein-Maxwell theory along the lines proposed by Drummond and Hathrell for the linear electrodynamics [24]. Based on the results of the paper [25], we considered in [26, 27, 28] a three-parameter gauge-invariant non-minimal EYM model linear in curvature. Next natural step was a formulation of a non-minimal Einstein-Yang-Mills-Higgs (EYMH) theory, and this process, of course, also admits different approaches. Taking into account the Higgs scalar multiplet we follow, first, the ideas, proposed in [29, 30, 31, 32], second the concept of derivative coupling, developed in [33, 34, 35].

In this paper we establish a five-parameter non-minimal Einstein-Yang-Mills-Higgs model. The first three coupling parameters, q_1, q_2, q_3, describe a non-minimal interaction of Yang-Mills field and gravitational field. The fourth and fifth parameters, q_4, q_5, describe the so-called gauge-invariant non-minimal "derivative coupling" of the Higgs field with gravity. Since the gauge-invariant derivative, $\hat{D}_m\boldsymbol{\Phi}$, contains the potential of the Yang-Mills field, the corresponding non-minimal term is associated with "triple" interaction, namely, gravitational and scalar fields, gauge and scalar fields, and gauge and gravitational fields.

It is clear that curvature induced effects predicted by the non-minimal EYMH theory should be examined by the analysis of particle dynamics and wave propagation. Wave propagation in media is known to be described in the framework of effective metric formalism (see, e.g., [36, 37, 38] for the history details and references). The first metric from the class of effective ones, namely, the optical metric, was introduced by Gordon [39] for isotropic media. In the effective spacetime with optical metric light propagates as in vacuum, light rays follow geodesics [40] and the wave vector is a null vector of this metric. Written in terms of the optical metric, the constitutive equations, linking the excitation tensor and the Maxwell tensor in electrodynamics of isotropic media, have the same formal structure as for vacuum electrodynamics [41, 42]. The second representative of the class of effective metrics is the acoustic metric [36, 37, 38]. Using this metric, one can regard sound waves as quasi-particles moving in the effective spacetime (e.g., [43]).

The covariant Fresnel equation is known to be a straightforward way to obtain effective metrics in electrodynamics (see [42] and references therein). It appears as a product of the geometrical optics approach and can be reduced to one eikonal equation in the isotropic case. Another way is connected with algebraic analysis of the constitutive equations. In electrodynamics this approach is developed and successfully used by Hehl and Obukhov [42]. In particular, these authors obtained the Minkowski metric as a kind of "tensorial square root" of the fourth-rank constitutive tensor, linking the excitation tensor and Maxwell (field strength) tensor in the so-called premetric electrodynamics.

When the medium is anisotropic, light propagation is accompanied by the phenomenon of birefringence [44]. When anisotropy is of the uniaxial type the Fresnel equation introduces two effective (optical) Lorentzian metrics [45], which define two light cones [36, 42, 45]. In the biaxial anisotropic media the effective (optical) metrics are in general non-Lorentzian [45], but again the birefringence revives bi-metricity.

Another way (algebraic) yields the same result. As it was shown in [46], the fourth-rank constitutive tensor can be generally reconstructed out of two symmetric second rank tensor fields, thus introducing the so-called associated metrics. Besides, it was shown in [46], that the transition between different representations of the constitutive tensor in terms of different pairs of associated metrics is governed by invariance properties in an associated two-dimensional internal vector space. Despite the fact, that interpretation of these associated metrics as Lorentzian optical metrics is restricted to the uniaxial case, the representation of constitutive equations in terms of associated metrics is valid in linear electrodynamics of an arbitrary anisotropic medium (exceptional case relates to the presence of skewons [42]).

Here we suggest the generalization of the formalism of associated metrics for the case of gauge field and introduce a new type of effective metrics, the color metrics. In order to motivate this generalization, we consider here a specific sort of vacuum field configuration, indicated as vacuum interacting with curvature, or equivalently, non-minimal vacuum. The five-parameter non-minimal EYMH model, considered below, is a convenient model for our purpose due to two reasons. First, the non-minimal EYMH model admits a clear Lagrangian formulation, the master equations for gravitational, gauge and scalar fields being obtained by a direct variational procedure. Second, the non-minimal master equations look like the corresponding equations for the anisotropic medium. In other words, in the presence of the curvature coupling of the gravitational, gauge and scalar fields the vacuum non-minimal EYMH model can be reformulated as a minimal one, but in some effective medium (quasi-vacuum, or vacuum, interacting with spacetime curvature). As a result, the effective metrics for the Yang-Mills field (color metrics) and for the Higgs field (color-acoustic metric) appear in a natural way.

The paper is organized as follows. In Section II we introduce basic formalism of the EYMH theory. In Section III we obtain the non-minimal extensions of the master equations for the gauge, scalar and gravity fields in the

framework of five-parameter model. In Section IV we discuss the formalism of multi-metric representation of the constitutive equations for the Yang-Mills and Higgs fields, and apply this formalism to the case of uniaxial non-minimal vacuum. In Section V we study in detail the cosmological applications of the effective metric formalism for Bianchi-I, Friedmann-Lemaître-Robertson-Walker (FLRW) and de Sitter models. Conclusions summarize the results. In Appendix A we consider the symmetry properties of the non-minimal constitutive tensor for the Yang-Mills and Higgs fields. Appendix B contains the WKB-analysis of the problem, which explains, why the associated metrics for the Yang-Mills field can be treated as color metrics. In Appendix C we present the tidal force, which gives an alternative description of the particle motion in the presence of non-minimal (curvature induced) interactions.

II. PREAMBLE

Our aim is to consider the EYMH model with the action functional of the following type

$$S_{\text{(EYMH)}} = \int d^4x \sqrt{-g} \left\{ \frac{R + 2\Lambda}{\kappa} + \frac{1}{2} C^{ikmn}_{(a)(b)} F^{(a)}_{ik} F^{(b)}_{mn} - \mathcal{C}^{ik}_{(a)(b)} \hat{D}_i \Phi^{(a)} \hat{D}_k \Phi^{(b)} + V(\Phi^2) \right\} . \tag{1}$$

Here $g = \det(g_{ik})$ is the determinant of a metric tensor g_{ik}, R is the Ricci scalar, Λ is the cosmological constant, the multiplet of real tensor fields $F^{(a)}_{ik}$ describes the strength of gauge field, the symbol $\Phi^{(a)}$ denotes the multiplet of the Higgs scalar real fields, $\hat{D}_k$ denotes the gauge-invariant derivative, $V(\Phi^2)$ is a potential of the Higgs field, and $\Phi^2 \equiv (\Phi^{(a)} \Phi_{(a)})$. Latin indices without parentheses run from 0 to 3, (a) and (b) are the group indices, the summation over repeating indices is assumed. The quantities $C^{ikmn}_{(a)(b)}$ and $\mathcal{C}^{ik}_{(a)(b)}$ denote the so-called constitutive tensors for the gauge and scalar fields, respectively. They contain neither Yang-Mills strength tensor $F^{(a)}_{ik}$, nor gauge-covariant derivative of the Higgs field $\hat{D}_k \Phi^{(b)}$, i.e., the Lagrangian in (1) is quadratic in these quantities. Since the tensors $C^{ikmn}_{(a)(b)}$ and $\mathcal{C}^{ik}_{(a)(b)}$ contain group indices (a) and (b), the metric in the group space $G_{(a)(b)}$, the scalar fields $\Phi^{(a)}$ and some additional directors in the group space, $q^{(a)}$, can be used to reconstruct these tensors. The symmetry properties of the tensors $C^{ikmn}_{(a)(b)}$ and $\mathcal{C}^{ik}_{(a)(b)}$ and their general decompositions are considered in detail in Appendix A. For the $U(1)$ symmetry such theory is discussed in [25, 47], here we consider $SU(n)$ $(n > 1)$ models. Let us mention that in this paper we restrict ourselves by the condition that only the metric tensor in the group space, $G_{(a)(b)}$, is used to construct the tensors $C^{ikmn}_{(a)(b)}$ and $\mathcal{C}^{ik}_{(a)(b)}$. Other constructions, for instance, $\Phi_{(a)} \Phi_{(b)}$ tensor, etc., are considered in [48].

In addition to the real fields $F^{(a)}_{ik}$, $\Phi^{(a)}$ and $A^{(a)}_i$ (the Yang-Mills field potential) we use below the symbols $\mathbf{F}_{ik}$, $\mathbf{\Phi}$ and $\mathbf{A}_i$. In literature there are few alternative definitions of these quantities, despite all versions give exactly the same result in terms of multiplets of real fields. In order to avoid the ambiguity, we stress, that in this paper we follow the definitions of the book [49] (see Section 4.3.) Thus, we consider the Yang-Mills field $\mathbf{F}_{mn}$ and the Higgs field $\mathbf{\Phi}$ taking values in the Lie algebra of the gauge group $SU(n)$ (adjoint representation):

$$\mathbf{F}_{mn} = -i\mathcal{G}\mathbf{t}_{(a)} F^{(a)}_{mn}, \quad \mathbf{A}_m = -i\mathcal{G}\mathbf{t}_{(a)} A^{(a)}_m, \quad \mathbf{\Phi} = \mathbf{t}_{(a)} \Phi^{(a)} . \tag{2}$$

Here $\mathbf{t}_{(a)}$ are the Hermitian traceless generators of $SU(n)$ group, thus, $\mathbf{\Phi}$ is considered to be Hermitian, but $\mathbf{F}_{mn}$ and $\mathbf{A}_i$ are anti-Hermitian. The group index (a) runs from 1 to $n^2 - 1$. The scalar products of the Yang-Mills and Higgs fields (indicated by bold letters) are defined in terms of the traces of the corresponding matrices (see, [49]), the scalar product of the generators $\mathbf{t}_{(a)}$ and $\mathbf{t}_{(b)}$ is chosen to be equal to:

$$\left(\mathbf{t}_{(a)}, \mathbf{t}_{(b)} \right) \equiv 2\text{Tr } \mathbf{t}_{(a)}\mathbf{t}_{(b)} \equiv G_{(a)(b)} . \tag{3}$$

The symmetric tensor $G_{(a)(b)}$ plays a role of a metric in the group space and the generators can be chosen so that the metric is equal to the Kronecker delta. The representation (2), (3) allows to consider the multiplets of the real fields $\{F^{(a)}_{mn}\}$, $\{A^{(a)}_i\}$ and $\{\Phi^{(a)}\}$ as components of the corresponding vectors in the $n^2 - 1$ dimensional group space. The operations with the group indices (a) are assumed to be the following: the repeating indices denote the convolution, and the rule $\Phi_{(a)} = G_{(a)(b)} \Phi^{(b)}$ for the indices lowering takes place. In such terms the gauge invariants in the action functional (15) is reduced to

$$\Phi^2 \equiv (\mathbf{\Phi}, \mathbf{\Phi}) \Rightarrow \Phi_{(a)} \Phi^{(a)}, \quad (\mathbf{F}_{mn}, \mathbf{F}^{mn}) \Rightarrow -\mathcal{G}^2 F^{(a)}_{mn} F^{mn}_{(a)} ,$$

$$\left(\hat{D}_m \mathbf{\Phi}, \hat{D}^m \mathbf{\Phi} \right) \Rightarrow \hat{D}_m \Phi^{(a)} \hat{D}^m \Phi_{(a)} . \tag{4}$$

The Yang-Mills fields $F_{mn}^{(a)}$ are connected with the potentials of the gauge field $A_i^{(a)}$ by the well-known formulas (see, e.g., [49, 50, 51, 52])

$$\mathbf{F}_{mn} = \nabla_m \mathbf{A}_n - \nabla_n \mathbf{A}_m + [\mathbf{A}_m, \mathbf{A}_n] \Rightarrow F_{mn}^{(a)} = \nabla_m A_n^{(a)} - \nabla_n A_m^{(a)} + \mathcal{G} f_{(b)(c)}^{(a)} A_m^{(b)} A_n^{(c)} . \tag{5}$$

Here ∇_m is a covariant spacetime derivative, the symbols $f_{(b)(c)}^{(a)}$ denote the real structure constants of the gauge group $SU(n)$. The gauge covariant derivative $\hat{D}_m \mathbf{\Phi} \equiv \mathbf{t}_{(a)} \hat{D}_m \Phi^{(a)}$ is defined according to the formulas ([49], Eqs.(4.46, 4.47))

$$\hat{D}_m \mathbf{\Phi} \equiv \nabla_m \mathbf{\Phi} + [\mathbf{A}_m, \mathbf{\Phi}] \Rightarrow \hat{D}_m \Phi^{(a)} \equiv \nabla_m \Phi^{(a)} + \mathcal{G} f_{(b)(c)}^{(a)} A_m^{(b)} \Phi^{(c)} . \tag{6}$$

For the derivative of arbitrary tensor defined in the group space we use the following rule [51]:

$$\hat{D}_m Q_{\cdots(d)}^{(a)\cdots} \equiv \nabla_m Q_{\cdots(d)}^{(a)\cdots} + \mathcal{G} f_{\cdot(b)(c)}^{(a)} A_m^{(b)} Q_{\cdots(d)}^{(c)\cdots} - \mathcal{G} f_{\cdot(b)(d)}^{(c)} A_m^{(b)} Q_{\cdots(c)}^{(a)\cdots} + \dots . \tag{7}$$

The definition of the commutator in (5) and (6) is based on the relation

$$\left[\mathbf{t}_{(a)}, \mathbf{t}_{(b)}\right] = i f_{(a)(b)}^{(c)} \mathbf{t}_{(c)} , \tag{8}$$

providing the formula

$$f_{(c)(a)(b)} \equiv G_{(c)(d)} f_{(a)(b)}^{(d)} = -2i \; \mathrm{Tr} \; \left[\mathbf{t}_{(a)}, \mathbf{t}_{(b)}\right] \mathbf{t}_{(c)} . \tag{9}$$

The structure constants $f_{(a)(b)(c)}$ are supposed to be antisymmetric under exchange of any two indices [49, 50, 51]. Metric $G_{(a)(b)}$ and the structure constants $f_{(a)(c)}^{(d)}$ are supposed to be constant tensors in the standard and covariant manner [51]. This means that

$$\partial_m G_{(a)(b)} = 0 , \quad \hat{D}_m G_{(a)(b)} = 0 , \quad \partial_m f_{(b)(c)}^{(a)} = 0 , \quad \hat{D}_m f_{(b)(c)}^{(a)} = 0 . \tag{10}$$

Furthermore, when the basis $\mathbf{t}_{(a)}$ is chosen to provide the relation $G_{(a)(b)} = \delta_{(a)(b)}$, it holds :

$$\frac{1}{n} f_{(a)(c)}^{(d)} f_{(d)(b)}^{(c)} = \delta_{(a)(b)} = G_{(a)(b)} \tag{11}$$

and

$$\left\{\mathbf{t}_{(a)}, \mathbf{t}_{(b)}\right\} \equiv \mathbf{t}_{(a)} \mathbf{t}_{(b)} + \mathbf{t}_{(b)} \mathbf{t}_{(a)} = \frac{1}{n} \delta_{(a)(b)} \mathbf{I} + d_{(a)(b)}^{(c)} \mathbf{t}_{(c)} \tag{12}$$

[51] with the completely symmetric coefficients $d_{(c)(a)(b)}$ ($\mathbf{I}$ is the matrix-unity). The tensor $F_{(a)}^{ik}$ satisfies the relation

$$\hat{D}_k {}^* F_{(a)}^{ik} = 0 , \tag{13}$$

the asterisk introduces the dual tensor

$$^* F_{(a)}^{ik} = \frac{1}{2} \epsilon^{ikls} F_{ls(a)} , \tag{14}$$

where $\epsilon^{ikls} = \frac{1}{\sqrt{-g}} E^{ikls}$ is the Levi-Civita tensor, E^{ikls} is the completely antisymmetric symbol with $E^{0123} = -E_{0123} = 1$.

As a first step we consider here the so-called non-minimal EYMH model, for which the constitutive tensors are constructed using the spacetime and group metrics, the Riemann and Ricci tensors, and Ricci scalar. Other possibilities will be studied in future papers.

III. FIVE-PARAMETER NON-MINIMAL EYMH MODEL

A. Non-minimal action functional

In the context of this paper we assume, that the constitutive tensors $C_{(a)(b)}^{ikmn}$ and $\mathcal{C}_{(a)(b)}^{ik}$, included into the action functional (1), contain neither contributions from the gauge field ($A_i^{(a)}$, $F_{mn}^{(a)}$) nor contributions from the Higgs field ($\Phi^{(a)}$, $\nabla_k \Phi^{(a)}$, etc.). The tensors $C_{(a)(b)}^{ikmn}$ and $\mathcal{C}_{(a)(b)}^{ik}$ can be generally reconstructed using tensor quantities of several types. The quantities of the first type do not contain derivatives of the metric; they include the metric g_{ik} itself, timelike four-vector U^k of the macroscopic velocity of the Yang-Mills-Higgs system as a whole, spacelike four-vectors $\mathcal{D}_{(\alpha)}^k$ directed along anisotropy axes, tensors in the group space, such as $G_{(a)(b)}$, $f_{(b)(c)}^{(a)}$, ... etc. Tensor quantities of the second type contain the first derivative of the metric only, they can be rewritten in terms of covariant derivatives $\nabla_m U^k$ and $\nabla_m \mathcal{D}_{(\alpha)}^k$. The tensor quantities of the third type consist of second derivatives of the metric and can be represented in terms of the Riemann tensor $R^i{}_{kmn}$ and its convolutions: the Ricci tensor $R_{kn} \equiv R^m{}_{kmn}$ and the Ricci scalar $R \equiv R_k^k$. The quantities of the next type may include the covariant derivative of the Riemann tensor $\nabla_j R_{ikmn}$, the tensor $R^{ikmn}\nabla_m U_n$, etc. When the terms containing the velocity four-vector U^i and its derivatives are taken into account, one deals with the so-called *dynamo*-phenomena (see, e.g., [44] for dynamo-optical effects). When the four-vectors $\mathcal{D}_{(\alpha)}^k$ and its derivatives are inserted into the Lagrangian, one assumes that the system contains a spatially anisotropic material subsystem. One refers to the effects induced by curvature, when the Lagrangian contains the Riemann tensor and its convolutions, and there are no contributions from U^k and $\mathcal{D}_{(\alpha)}^k$.

In this paper we focus on effects induced by curvature and consider the Lagrangian of the EYMH theory *linear* in the Riemann tensor. The ansatz of linearity with respect to Riemann tensor leads us to the five-parameter model with the action functional written in the following (generic) form

$$S_{\text{(NMEYMH)}} = \int d^4x \sqrt{-g} \ \left\{ \frac{R + 2\Lambda}{\kappa} + \frac{1}{2} F_{ik}^{(a)} F_{(a)}^{ik} - \hat{D}_m \Phi^{(a)} \hat{D}^m \Phi_{(a)} + m^2 \Phi^2 \right. $$
$$\left. + \frac{1}{2}\mathcal{R}^{ikmn} F_{ik}^{(a)} F_{mn(a)} - \Re^{mn} \hat{D}_m \Phi^{(a)} \hat{D}_n \Phi_{(a)} \right\} , \tag{15}$$

where the so-called susceptibility tensors $\mathcal{R}^{ikmn}$ and $\Re^{mn}$ are defined as follows:

$$\mathcal{R}^{ikmn} \equiv \frac{q_1}{2} R \left(g^{im}g^{kn} - g^{in}g^{km} \right) + \frac{q_2}{2} (R^{im}g^{kn} - R^{in}g^{km} + R^{kn}g^{im} - R^{km}g^{in}) + q_3 R^{ikmn} , \tag{16}$$

$$\Re^{mn} \equiv q_4 R g^{mn} + q_5 R^{mn} . \tag{17}$$

This action describes the five-parameter non-minimal EYMH model, and q_1, q_2, ... q_5 are the constants of non-minimal coupling. The action (15) is the particular case of (1) with

$$C_{(a)(b)}^{ikmn} = \left[\frac{1}{2}(g^{im}g^{kn} - g^{in}g^{km}) + \mathcal{R}^{ikmn} \right] G_{(a)(b)} , \tag{18}$$

and

$$\mathcal{C}_{(a)(b)}^{mn} = G_{(a)(b)} \left(g^{mn} + \Re^{mn} \right) . \tag{19}$$

Below we indicate the tensor

$$\tilde{g}^{ik} = g^{ik} + \Re^{ik} \tag{20}$$

as a color-acoustic metric.

REMARK 1
The non-minimal susceptibility tensors $\mathcal{R}^{ikmn}$ (16) and $\Re^{mn}$ (17) can be rewritten in terms of irreducible parts of the curvature tensor

$$\mathcal{R}^{ikmn} = \lambda_1 \mathcal{G}^{ikmn} + \lambda_2 \mathcal{E}^{ikmn} + \lambda_3 \mathcal{C}^{ikmn} , \quad \Re^{mn} \equiv \lambda_4 R g^{mn} + \lambda_5 \mathcal{S}^{mn} \tag{21}$$

where $\mathcal{C}^{ikmn}$ is the traceless Weyl tensor

$$\mathcal{C}^{ikmn} \equiv R^{ikmn} - \mathcal{E}^{ikmn} - \mathcal{G}^{ikmn} , \quad \mathcal{C}^m{}_{nmk} = 0 , \tag{22}$$

and

$$\mathcal{E}^{ikmn} \equiv \frac{1}{2}\left(\mathcal{S}^{im}g^{kn}-\mathcal{S}^{in}g^{km}+\mathcal{S}^{kn}g^{im}-\mathcal{S}^{km}g^{in}\right)\right), \quad \mathcal{S}^{mn} \equiv R^{mn}-\frac{1}{4}Rg^{mn},$$

$$\mathcal{G}^{ikmn} \equiv \frac{1}{12}R\left(g^{im}g^{kn}-g^{in}g^{km}\right), \tag{23}$$

(we use the notations from the book [53]). The phenomenological parameters λ_1, λ_2, λ_3, λ_4 and λ_5 are connected with q_1, q_2, q_3, q_4 and q_5 by the linear relations

$$\lambda_1 = 6q_1+3q_2+q_3, \quad \lambda_2 = q_2+q_3, \quad \lambda_3 = q_3, \quad \lambda_4 = q_4+\frac{1}{4}q_5, \quad \lambda_5 = q_5. \tag{24}$$

The representations (16) and (17) are equivalent to (21) from a computational point of view, nevertheless, we use below the decomposition (16) taking into account some historical motivation (see, e.g., reviews [19, 20]). In particular, the phenomenological model proposed by Prasanna in [54] contains the Riemann tensor only ($q_1 = q_2 = q_4 = q_5 = 0$) and gives a first example of the violation of the strong equivalence principle in the framework of non-minimal electrodynamics. The first non-phenomenological model elaborated by Drummond and Hathrell in [24] is based on one-loop corrections to quantum electrodynamics in curved spacetime; the coupling parameters are calculated directly in [24] yielding $q_1 = -5q$, $q_2 = 13q$, $q_3 = -2q$ (or equivalently $\lambda_1 = 7q$, $\lambda_2 = 11q$, $\lambda_3 = -2q$). The positive parameter q, $q \equiv \frac{\alpha\lambda_e^2}{180\pi}$ is constructed by using the fine structure constant α, and the Compton wavelength of the electron λ_e. The models investigated by Horndeski [22, 55], by Müller-Hoissen [23], by Müller-Hoissen and Sippel [56], contain master equations of the second order only; they are characterized by the following values of the coupling parameters: $q_1 = -q$, $q_2 = 2q$, $q_3 = -q$ (or equivalently $\lambda_1 = -q$, $\lambda_2 = q$, $\lambda_3 = -q$) (here parameter q is arbitrary). These relations can be recovered by the ansatz that the non-minimal susceptibility tensor $\mathcal{R}_{ikmn}$ is proportional to the double dual Riemann tensor ${}^*R^*_{ikmn}$, i.e., $\mathcal{R}_{ikmn} = \gamma\,{}^*R^*_{ikmn}$, for some γ [25]. Analogously, one can use the Weyl tensor $\mathcal{C}_{ikmn}$ in the relation $\mathcal{R}_{ikmn} = \omega\,\mathcal{C}_{ikmn}$, for some ω [25]. Thus, there are different physical and mathematical motives to specify coupling constants, but here we prefer to consider generic q_1, q_2, ... q_5, introduced phenomenologically.

REMARK 2
The Lagrangian of the non-minimal model linear in curvature can contain also the term $\xi R\Phi^{(a)}\Phi_{(a)}$ without gradients of the Higgs fields, the corresponding model with $\xi \neq 0$ is the six-parameter one (see, e.g., [21]). Here we focus on five-parameter EYMH model only.

REMARK 3
Let us summarize the assumptions, which we made in the formulation of the five-parameter EYMH model.
(i) The Lagrangian of the EYMH model is quadratic in $F_{mn}^{(a)}$ and $\hat{D}_m\Phi^{(a)}$ (see (1)).
(ii) The Lagrangian of the EYMH model is linear in the curvature tensor (see (15)-(17)).
(iii) The group indices in (15) are coupled by the metric $G_{(a)(b)}$ only.
   The latter assumption leads to an important consequence: it provides all the constitutive tensors to be multiplicative, i.e., they are products of the metric in the group space and the corresponding tensors defined in the spacetime. In its turn, this consequence guarantees, that for the five-parameter EYMH model the constitutive tensors are symmetric with respect to transposition of the group indices.

B.   Non-minimal extension of the Yang-Mills equations

The variation of the action $S_{\text{(NMEYMH)}}$ with respect to the Yang-Mills potential $A_i^{(a)}$ yields

$$\hat{D}_k\mathcal{H}_{(a)}^{ik} = -\mathcal{G}(\hat{D}_k\Phi^{(b)})f_{(a)(b)(c)}\Phi^{(c)}\left(g^{ik}+\Re^{ik}\right), \tag{25}$$

where the tensor $\mathcal{H}_{(a)}^{ik}$ is defined as

$$\mathcal{H}_{(a)}^{ik} = \left[\frac{1}{2}(g^{im}g^{kn}-g^{in}g^{km})+\mathcal{R}^{ikmn}\right]G_{(a)(b)}F_{mn}^{(b)}. \tag{26}$$

Equivalently, one can write

$$\hat{D}_k\mathbf{H}^{ik} \equiv \nabla_k\mathbf{H}^{ik}+\left[\mathbf{A}_k,\mathbf{H}^{ik}\right] = \mathcal{G}^2\left[\left(g^{ik}+\Re^{ik}\right)\hat{D}_k\mathbf{\Phi},\mathbf{\Phi}\right], \tag{27}$$

where

$$\mathbf{H}^{ik} = \mathbf{F}^{ik} + \mathcal{R}^{ikmn}\mathbf{F}_{mn}\,. \tag{28}$$

By analogy with electrodynamics of continuous media the tensor $\mathbf{H}^{ik}$ in the non-minimal Yang-Mills equations can be called excitation tensor. Below we use the term "color excitation". Equation (26) is in fact a linear constitutive law, connecting the color excitation tensor $H_{(a)}^{ik}$ and the field strength tensor $F_{mn}^{(b)}$ (see, e.g., [42, 44, 57] for the $U(1)$ symmetry). It can be clearly rewritten as

$$\mathcal{H}_{(a)}^{ik} = C_{(a)(b)}^{ikmn}\ F_{mn}^{(b)}\,, \tag{29}$$

demonstrating that the quantity $\mathcal{R}^{ikmn}G_{(a)(b)}$ plays the role of color susceptibility tensor. The term $\mathcal{G}^2\left(g^{ik} + \Re^{ik}\right)\left[\hat{D}_k\mathbf{\Phi}, \mathbf{\Phi}\right]$ plays, respectively, the role of color current induced by the Higgs fields.

C. Non-minimal extension of the Higgs field equations

The variation of the action $S_{(\mathrm{NMEYMH})}$ with respect to the Higgs scalar field $\Phi^{(a)}$ yields

$$\hat{D}_m\left[(g^{mn} + \Re^{mn})\,\hat{D}_n\mathbf{\Phi}\right] = -m^2\mathbf{\Phi}\,. \tag{30}$$

By analogy with thermodynamics the four-vector $\hat{D}_n\mathbf{\Phi}$ can be indicated as color four-gradient of the Higgs field (the analog of the temperature three-gradient), and the four-vector $\mathbf{\Psi}^m$ can be called the color flux four-vector (the analog of the heat flux). Thus, the relation

$$\mathbf{\Psi}^m = (g^{mn} + \Re^{mn})\,\hat{D}_n\mathbf{\Phi} \tag{31}$$

is in fact a non-minimal constitutive law, generalizing the Fourier law in thermodynamics, its general form being

$$\Psi_{(a)}^m = \mathcal{C}_{(a)(b)}^{mn}(\hat{D}_n\mathbf{\Phi})^{(b)}\,. \tag{32}$$

D. Master equations for the gravitational field

In the non-minimal theory linear in curvature, the equations for the gravity field related to the action functional $S_{(\mathrm{NMEYMH})}$ take the form

$$\left(R_{ik} - \frac{1}{2}Rg_{ik}\right) = \Lambda g_{ik} + \kappa T_{ik}^{(NMYMH)}\,. \tag{33}$$

The principal novelty of these equations in comparison with the well-known equations for non-minimal scalar field is associated with the third, fourth, etc., terms in the decomposition

$$T_{ik}^{(NMYMH)} = T_{ik}^{(YM)} + T_{ik}^{(H)} + q_1 T_{ik}^{(I)} + q_2 T_{ik}^{(II)} + q_3 T_{ik}^{(III)} + q_4 T_{ik}^{(IV)} + q_5 T_{ik}^{(V)}\,. \tag{34}$$

The first term $T_{ik}^{(YM)}$:

$$T_{ik}^{(YM)} \equiv \frac{1}{4}g_{ik}F_{mn}^{(a)}F_{(a)}^{mn} - F_{in}^{(a)}F_{k\ (a)}^{\ n}\,, \tag{35}$$

is a stress-energy tensor of pure Yang-Mills field. The second one, $T_{ik}^{(H)}$,

$$T_{ik}^{(H)} = \hat{D}_i\Phi^{(a)}\hat{D}_k\Phi_{(a)} - \frac{1}{2}g_{ik}\hat{D}_m\Phi^{(a)}\hat{D}^m\Phi_{(a)} + \frac{1}{2}m^2\Phi^2\,g_{ik} \tag{36}$$

is a stress-energy tensor of the Higgs field. The definitions of other five tensors are related to the corresponding coupling constants $q_1, q_2, \ldots, q_5$:

$$T_{ik}^{(I)} = R\,T_{ik}^{(YM)} - \frac{1}{2}R_{ik}F_{mn}^{(a)}F_{(a)}^{mn} + \frac{1}{2}\left[\hat{D}_i\hat{D}_k - g_{ik}\hat{D}^l\hat{D}_l\right]\left[F_{mn}^{(a)}F_{(a)}^{mn}\right]\,, \tag{37}$$

$$T_{ik}^{(II)} = -\frac{1}{2}g_{ik}\left[\hat{D}_m\hat{D}_l\left(F^{mn(a)}F^l_{\ n(a)}\right) - R_{lm}F^{mn(a)}F^l_{\ n(a)}\right]$$

$$-F^{ln(a)}\left(R_{il}F_{kn(a)} + R_{kl}F_{in(a)}\right) - R^{mn}F_{im}^{(a)}F_{kn(a)} - \frac{1}{2}\hat{D}^m\hat{D}_m\left(F_{in}^{(a)}F_{k\ (a)}^{\ n}\right)$$

$$+\frac{1}{2}\hat{D}_l\left[\hat{D}_i\left(F_{kn}^{(a)}F_{(a)}^{ln}\right) + \hat{D}_k\left(F_{in}^{(a)}F_{(a)}^{ln}\right)\right], \tag{38}$$

$$T_{ik}^{(III)} = \frac{1}{4}g_{ik}R^{mnls}F_{mn}^{(a)}F_{ls(a)} - \frac{3}{4}F^{ls(a)}\left(F_{i\ (a)}^{\ n}R_{knls} + F_{k\ (a)}^{\ n}R_{inls}\right)$$

$$-\frac{1}{2}\hat{D}_m\hat{D}_n\left[F_i^{\ n(a)}F_{k\ (a)}^{\ m} + F_k^{\ n(a)}F_{i\ (a)}^{\ m}\right], \tag{39}$$

$$T_{ik}^{(IV)} = \left(R_{ik} - \frac{1}{2}Rg_{ik}\right)\hat{D}_m\Phi^{(a)}\hat{D}^m\Phi_{(a)} + R\,\hat{D}_i\Phi^{(a)}\hat{D}_k\Phi_{(a)}$$

$$+\left(g_{ik}\hat{D}_n\hat{D}^n - \hat{D}_i\hat{D}_k\right)\left[\hat{D}_m\Phi^{(a)}\hat{D}^m\Phi_{(a)}\right], \tag{40}$$

$$T_{ik}^{(V)} = \hat{D}_m\Phi^{(a)}\left[R_i^m\hat{D}_k\Phi_{(a)} + R_k^m\hat{D}_i\Phi_{(a)}\right] - \frac{1}{2}R_{ik}\,\hat{D}_m\Phi^{(a)}\hat{D}^m\Phi_{(a)}$$

$$+\frac{1}{2}g_{ik}\hat{D}_m\hat{D}_n\left[\hat{D}^m\Phi^{(a)}\hat{D}^n\Phi_{(a)}\right]$$

$$-\frac{1}{2}\hat{D}^m\left\{\hat{D}_i\left[\hat{D}_m\Phi^{(a)}\hat{D}_k\Phi_{(a)}\right] + \hat{D}_k\left[\hat{D}_m\Phi^{(a)}\hat{D}_i\Phi_{(a)}\right] - \hat{D}_m\left[\hat{D}_i\Phi^{(a)}\hat{D}_k\Phi_{(a)}\right]\right\}. \tag{41}$$

The Einstein tensor $G_{ik} \equiv R_{ik} - \frac{1}{2}g_{ik}R$ is the divergence-free one, thus, the tensor $T_{ik}^{(NMYMH)}$ in the right-hand-side of (33) has to satisfy the differential condition

$$\nabla^k T_{ik}^{(NMYMH)} = 0. \tag{42}$$

One can prove that it is valid automatically, when $F_{ik}^{(a)}$ is a solution of the Yang-Mills equations (27), and $\Phi^{(a)}$ satisfy the Higgs equations (30). In order to check this fact directly, one has to use the Bianchi identities and the properties of the Riemann tensor:

$$\nabla_i R_{klmn} + \nabla_l R_{ikmn} + \nabla_k R_{limn} = 0, \quad R_{klmn} + R_{mkln} + R_{lmkn} = 0, \tag{43}$$

as well as the rules for the commutation of covariant derivatives

$$(\nabla_l\nabla_k - \nabla_k\nabla_l)\mathcal{A}^i = \mathcal{A}^m R^i_{\cdot mlk}, \tag{44}$$

(this rule is written here for vectors only). The procedure of checking is analogous to the one, described in [58] and we omit it here.

E.   Interpretation of the gravity field equations in terms of gravitational excitation and strength

Since the Riemann tensor $R^i{}_{klm}$ can be considered as a gravitational field strength, an analog of F_{ik} in electrodynamics (see, e.g., [59]), one can emphasize an interesting analogy with electrodynamics. Let us mention that one of the ingredients of electrodynamics of media is a linear constitutive law

$$H^{im} = \mathcal{H}^{im} + C^{imls} F_{ls}\,, \tag{45}$$

where H^{im} is an excitation tensor, C^{imls} is a constitutive tensor and $\mathcal{H}^{im}$ is a spontaneous polarization-magnetization tensor, which does not depend on the Maxwell tensor [57]. The Maxwell tensor and the excitation tensor satisfy equations

$$\nabla_i F_{kl} + \nabla_l F_{ik} + \nabla_k F_{li} = 0\,, \quad \nabla_k H^{ik} = -\frac{4\pi}{c} I^i\,, \tag{46}$$

where I^i is a four-vector of the current of unbounded charges. In analogy with (45) we can introduce a linear constitutive equation for the gravity field

$$Z_{imkn} = \mathcal{H}_{imkn} + \mathcal{C}_{imkn}{}^{pqls} R_{pqls}\,, \tag{47}$$

where Z_{imkn} is a tensor of gravitational excitation, $\mathcal{C}_{imkn}{}^{pqls}$ is a constitutive tensor, and $\mathcal{H}_{imkl}$ is a tensor of gravitational polarization. In analogy with (46) the field strength R_{pqls} satisfies the Bianchi identity (43). The Einstein equations (33) with (34) look like the convolution of the constitutive equations (47) with g^{mn}, if we put

$$\mathcal{C}_{imkn}{}^{pqls} = \frac{1}{2}\left[\mathcal{L}_{ik}{}^{([pq][ls])}g_{mn} - \mathcal{L}_{in}{}^{([pq][ls])}g_{mk} + \mathcal{L}_{mn}{}^{([pq][ls])}g_{ik} - \mathcal{L}_{mk}{}^{([pq][ls])}g_{in}\right]$$

$$-\frac{1}{6}\left(g_{ik}g_{mn} - g_{in}g_{mk}\right)g^{rt}\mathcal{L}_{rt}{}^{([pq][ls])}\,,$$

$$\mathcal{L}_{ik}{}^{pqls} \equiv g^{qs}\left[\delta_i^p \delta_k^l - \frac{1}{2}g^{pl}g_{ik}\right] + \frac{3}{4}\kappa q_3\left[F_{(a)}^{ls}\left(\delta_i^p F_k^{(a)q} + \delta_k^p F_i^{(a)q}\right) - \frac{1}{3}g_{ik}F_{(a)}^{pq}F^{ls(a)}\right]$$

$$+\kappa g^{qs}\left\{\frac{1}{2}\delta_i^p \delta_k^l\left[q_1 F_{jh}^{(a)}F_{(a)}^{jh} + (q_5 - 2q_4)\hat{D}_j\Phi^{(a)}\hat{D}^j\Phi_{(a)}\right]\right.$$

$$-g^{pl}\left[q_1 T_{ik}^{(YM)} + q_4\left(T_{ik}^{(H)} - \frac{1}{2}m^2\Phi^2 g_{ik}\right)\right] - q_5 \hat{D}^l\Phi^{(a)}\left[\delta_i^p \hat{D}_k\Phi_{(a)} + \delta_k^p \hat{D}_i\Phi_{(a)}\right]$$

$$\left.+q_2\left[-\frac{1}{2}g_{ik}F^{p(a)}_{j}F_{(a)}^{lj} + F_{(a)}^{lj}\left(\delta_i^p F_{kj}^{(a)} + \delta_k^p F_{ij}^{(a)}\right) + F_{i(a)}^{p}F_k^{(a)l}\right]\right\} \tag{48}$$

for the constitutive tensor, and

$$\mathcal{H}_{imkn} = \kappa\left(\mathcal{H}_{i[n}^{(G)}g_{k]m} - \mathcal{H}_{m[n}^{(G)}g_{k]i} - \frac{1}{3}\mathcal{H}_{ls}^{(G)}g^{ls}g_{i[n}g_{k]m}\right)\,, \tag{49}$$

$$\mathcal{H}_{ik}^{(G)} = \Lambda g_{ik} + T_{ik}^{(YM)} + T_{ik}^{(H)} + \frac{1}{2}q_1\left[\hat{D}_i\hat{D}_k - g_{ik}\hat{D}^l\hat{D}_l\right]\left[F_{mn}^{(a)}F_{(a)}^{mn}\right]$$

$$-\frac{1}{2}q_2\left\{g_{ik}\left[\hat{D}_m\hat{D}_l\left(F^{mn(a)}F^l{}_{n(a)}\right)\right] + \hat{D}^m\hat{D}_m\left(F_{in}^{(a)}F_k{}^n{}_{(a)}\right)\right.$$

$$\left.-\hat{D}_l\left[\hat{D}_i\left(F_{kn}^{(a)}F_{(a)}^{ln}\right) + \hat{D}_k\left(F_{in}^{(a)}F_{(a)}^{ln}\right)\right]\right\}$$

$$-\frac{1}{2}q_3\hat{D}_m\hat{D}_n\left[F_i^{\ n(a)}F_k^{\ m}{}_{(a)}+F_k^{\ n(a)}F_i^{\ m}{}_{(a)}\right]+q_4\left(g_{ik}\hat{D}_n\hat{D}^n-\hat{D}_i\hat{D}_k\right)\left[\hat{D}_m\Phi^{(a)}\hat{D}^m\Phi_{(a)}\right]$$

$$-\frac{1}{2}q_5\left\{\hat{D}^m\left\{\hat{D}_i\left[\hat{D}_m\Phi^{(a)}\hat{D}_k\Phi_{(a)}\right]+\hat{D}_k\left[\hat{D}_m\Phi^{(a)}\hat{D}_i\Phi_{(a)}\right]-\hat{D}_m\left[\hat{D}_i\Phi^{(a)}\hat{D}_k\Phi_{(a)}\right]\right\}\right.$$

$$\left.-g_{ik}\hat{D}_m\hat{D}_n\left[\hat{D}^m\Phi^{(a)}\hat{D}^n\Phi_{(a)}\right]\right\}\tag{50}$$

for the gravitational excitation. Here we use the following standard definitions for symmetrization and antisymmetrization of indices, respectively:

$$\mathcal{K}_{(mn)}\equiv\frac{1}{2}\left(\mathcal{K}_{mn}+\mathcal{K}_{nm}\right),\quad\mathcal{K}_{[mn]}\equiv\frac{1}{2}\left(\mathcal{K}_{mn}-\mathcal{K}_{nm}\right).\tag{51}$$

Finally, the identity

$$\nabla^k\left\{R_{ik}-\frac{1}{2}g_{ik}g^{mn}R_{mn}-\kappa T_{ik}^{(NMYMH)}-\Lambda g_{ik}\right\}=0\,,\tag{52}$$

written as

$$\nabla^k\left(Z_{imkn}g^{mn}\right)=0\,,\tag{53}$$

is an analog of the second Maxwell equation (46) with vanishing current. Clearly, the constitutive tensor (48) entering the constitutive law (47), is built using metric, quadratic combinations of the tensor $F_{ik}^{(a)}$ and color gradient of the Higgs field, and does not contain the derivatives of the spacetime metric.

Thus, in the presented five-parameter non-minimal model three constitutive tensors, $C_{(a)(b)}^{ikmn}$ (18), $\mathcal{C}_{(a)(b)}^{ik}$ (19) and $\mathcal{C}_{imkn}^{\quad\ pqls}$ (48), appear in a natural way. It can be considered as a motivation for the construction of more sophisticated models with the action functional (1).

IV. MULTI-METRIC REPRESENTATION OF THE CONSTITUTIVE TENSORS

A. General formalism

Our final purpose is to reconstruct effective metrics (associated, color and color-acoustic) for the five-parameter EYMH model with the action functional (15) for the one-axis Bianchi-I model, FLRW and de Sitter models. In order to explain the procedure of their finding for this five-parameter model, we consider, first, a general formalism (see next subsection and Apendices), then we reduce basic formulas to the case of uniaxial symmetry, and, finally, we obtain desired effective metrics. Let us mention, that we do not introduce new assumptions in addition to the three ones fixed above in the Remark 3, but we made a few simplifications, which follow from the symmetry of the corresponding cosmological model. For instance, the spacetime symmetry of the one-axis Bianchi-I model requires the symmetry of the constitutive tensors to be uni-axial with vanishing cross-effects. This is a reason, why we consider the appropriate reduced formulas instead of the general ones.

1. Decomposition of the tensor $C_{(a)(b)}^{ikmn}$

Let $g_{(a)}^{im(\alpha)}$ be tensor fields symmetric with respect to spacetime indices, i and m. Here $\alpha=1,2$ is the field number, and (a) is the group index (color number), taking only one value for $U(1)$ symmetry and running from 1 to n^2-1 for the $SU(n)$ symmetry. This Latin index is placed in parentheses. We restrict ourselves by the models with $C_{(a)(b)}^{ikmn}=C_{(b)(a)}^{ikmn}$, thus, the most general representation of this constitutive tensor in terms of $g_{(a)}^{im(\alpha)}$ is

$$C_{(a)(b)}^{ikmn}=\frac{1}{2\hat{\mu}}\sum_{(\alpha)(\beta)(c)(d)}G_{(\alpha)(\beta)(a)(b)}^{(c)(d)}\left(g_{(c)}^{im(\alpha)}\,g_{(d)}^{kn(\beta)}-g_{(c)}^{in(\alpha)}\,g_{(d)}^{km(\beta)}\right),\tag{54}$$

where the following symmetry of indices is assumed:

$$G^{(c)(d)}_{(\alpha)(\beta)(a)(b)} = G^{(c)(d)}_{(\alpha)(\beta)(b)(a)} = G^{(d)(c)}_{(\beta)(\alpha)(a)(b)} \,, \tag{55}$$

the sum being over all possible combinations of field numbers and color numbers. The factor $\hat{\mu}$ is introduced, as in [46], for convenience. The $C^{ikmn}_{(a)(b)}$ tensor has the general form (A4) (see Appendix A). It contains $21\frac{(n^2-1)n^2}{2}$ components. There are 10 components of $g^{im(\alpha)}_{(c)}$ for each α and (a). Thus, when we deal with two metrics, i.e., $\alpha = 1, 2$ for $n > 1$ one has $20(n^2-1)$ field components $g^{im(\alpha)}_{(a)}$, as well as $\frac{1}{2}(n^2-1)^2 n^2(2n^2-1)$ parameters (functions) $G^{(c)(d)}_{(\alpha)(\beta)(a)(b)}$, as a tool for reconstruction of $21\frac{(n^2-1)n^2}{2}$ components of the $C^{ikmn}_{(a)(b)}$ tensor. The reconstruction of $C^{ikmn}_{(a)(b)}$ along the line of (54) is always possible since the corresponding algebraic system is underdetermined. Let us mention that $U(1)$ case is a special one, and (see, e.g., [46]) one has 20 field components $g^{im(\alpha)}$, as well as 3 parameters $G_{(\alpha)(\beta)}$ to reconstruct 21 components of C^{ikmn}. In the case of $SU(2)$ symmetry we have, generally, 60 field components $g^{im(\alpha)}_{(a)}$, 126 components of $G^{(c)(d)}_{(\alpha)(\beta)(a)(b)}$ and 126 components of $C^{ikmn}_{(a)(b)}$. Thus, 60 parameters are arbitrary.

The quantity $G^{(c)(d)}_{(\alpha)(\beta)(a)(b)}$ is considered to be a twice covariant and twice contravariant tensor in the n^2-1 dimensional group space, the tensors $g^{im(\alpha)}_{(c)}$ are the vectors in that space. When α, i and m are fixed, the standard unitary transformations are assumed to conserve the scalar product $G^{(a)(b)}g^{im(\alpha)}_{(a)}g^{kn(\beta)}_{(b)}$. As well, when the spacetime indices and group indices are fixed, there exist a linear transformation from one set of $g^{im(\alpha)}_{(c)}$ to another one, and $g^{im(\alpha)}_{(c)}$ can be regarded as vectors in some effective two-dimensional space [46]. When we deal with the so-called "parallel fields" [60], i.e., when

$$g^{im(\alpha)}_{(a)} = q_{(a)}g^{im(\alpha)}\,, \quad G^{(a)(b)}q_{(a)}q_{(b)} = 1\,, \tag{56}$$

we can put

$$G^{(c)(d)}_{(\alpha)(\beta)(a)(b)} = G_{(\alpha)(\beta)}G_{(a)(b)}G^{(c)(d)}\,, \tag{57}$$

providing $C^{ikmn}_{(a)(b)}$ to have a multiplicative structure, $C^{ikmn}_{(a)(b)} = C^{ikmn}G_{(a)(b)}$. Thus, the model of parallel fields $g^{im(\alpha)}_{(a)}$ for the $SU(n)$ case covers the $U(1)$ model considered in [46]. When, on the contrary, we assume $C^{ikmn}_{(a)(b)} = C^{ikmn}G_{(a)(b)}$, the tensor $G^{(c)(d)}_{(\alpha)(\beta)(a)(b)}$ is not obligatory multiplicative, nevertheless, the parallel fields give one of the solutions of the reconstruction problem.

As any symmetric tensor, the quantities $g^{ik(\alpha)}_{(a)}$ can be decomposed with respect to their components parallel and orthogonal to the four-velocity U^i,

$$g^{ik(\alpha)}_{(a)} = \mathcal{B}^{(\alpha)}_{(a)}U^iU^k + \mathcal{D}^{i(\alpha)}_{(a)}U^k + \mathcal{D}^{k(\alpha)}_{(a)}U^i + \mathcal{S}^{ik(\alpha)}_{(a)}\,, \tag{58}$$

where

$$\mathcal{B}^{(\alpha)}_{(a)} = g^{ik(\alpha)}_{(a)}U_iU_k\,, \quad \mathcal{D}^{p(\alpha)}_{(a)} \equiv \Delta^p_i g^{ik(\alpha)}_{(a)}U_k\,, \quad \mathcal{S}^{pq(\alpha)}_{(a)} \equiv \Delta^p_i g^{ik(\alpha)}_{(a)}\Delta^q_k\,. \tag{59}$$

In the application of this formalism to cosmology (see below), one obtains directly that

$$\mathcal{D}^{p(\alpha)}_{(a)} = 0 \quad \Rightarrow \quad g^{ik(\alpha)}_{(a)} = U^iU^k + \mathcal{S}^{ik(\alpha)}_{(a)} \quad \Rightarrow \quad U_i g^{im(\alpha)}_{(a)} = U^m\,, \tag{60}$$

which means that the velocity four-vector U^k is the eigenvector for all tensors $g^{ik(\alpha)}_{(a)}$ with the eigenvalue equal to one.

2. Decomposition of the tensor $\mathcal{C}^{ik}_{(a)(b)}$

We assume that $\mathcal{C}^{ik}_{(a)(b)} = \mathcal{C}^{ik}_{(b)(a)} = \mathcal{C}^{ki}_{(b)(a)}$ and consider the following decomposition based on the effective metrics $\tilde{g}^{ik}_{(a)}$

$$\mathcal{C}^{ik}_{(a)(b)} = \Gamma^{(c)}_{(a)(b)}\tilde{g}^{ik}_{(c)}\,. \tag{61}$$

The tensor $\mathcal{C}^{ik}_{(a)(b)}$ is multiplicative, i.e., $\mathcal{C}^{ik}_{(a)(b)} = \mathcal{C}^{ik} G_{(a)(b)}$, and one can choose tensor $\tilde{g}^{ik}_{(c)}$ independent on group index and put

$$\mathcal{C}^{ik}_{(a)(b)} = G_{(a)(b)} \tilde{g}^{ik} . \tag{62}$$

This choice can be motivated by the case of parallel fields

$$\tilde{g}^{ik}_{(c)} = \tilde{g}^{ik} \, q_{(c)} , \tag{63}$$

for which the following representation is possible

$$\Gamma^{(c)}_{(a)(b)} q_{(c)} = G_{(a)(b)} . \tag{64}$$

The most general decomposition can be much more sophisticated, and we do not consider it here.

B. Effective metrics for the five-parameter EYMH model

1. General case

In the framework of the five-parameter non-minimal EYMH model the macroscopic tensor has multiplicative structure, i.e., $C^{ikmn}_{(a)(b)} = C^{ikmn} G_{(a)(b)}$, the color metrics $g^{ik(\alpha)}_{(a)}$ can be chosen independent on the group indices. One can say that in this case there is no "group multi-refringence", but the spacetime birefringence exists. Then, following [46] we represent the tensor $C^{ikmn}_{(a)(b)}$ (18) by the decomposition

$$C^{ikmn}_{(a)(b)} = G_{(a)(b)} \frac{1}{2\hat{\mu}} \left\{ \left[g^{im(A)} g^{kn(A)} - g^{in(A)} g^{km(A)} \right] \right.$$

$$\left. -\gamma \left[(g^{im(A)} - g^{im(B)})(g^{kn(A)} - g^{kn(B)}) - (g^{in(A)} - g^{in(B)})(g^{km(A)} - g^{km(B)}) \right] \right\} , \tag{65}$$

by choosing

$$G_{(A)(A)} + G_{(B)(A)} = 1 , \quad G_{(A)(B)} + G_{(B)(B)} = 0 , \quad \gamma \equiv G_{(A)(B)} = G_{(B)(A)} . \tag{66}$$

Here the indices (A) and (B) are no longer arbitrary: they are fixed in order to simplify the decomposition of the constitutive tensor [46]. We will refer below to this choice as to A-metric, $g^{im(A)}$, and B-metric, $g^{im(B)}$, as well as to the corresponding A-wave and B-wave. It is reasonable since generally the A-wave and B-wave are not obligatory "ordinary" and "extraordinary", as in classical optics.

With this decomposition the A-tensor, $g^{ik(A)}$, can be readily written as (see [46] for details)

$$g^{ik(A)} = U^i U^k + \hat{\mu} \varepsilon^{ik} . \tag{67}$$

The B-tensor, $g^{ik(B)}$, can be represented in terms of eigenvectors $X^i_{(1)}, X^i_{(2)}$, and $X^i_{(3)}$ of the permittivity tensor ε^{ik} [45]. These eigenvectors form the triad and possess the properties

$$g_{ik} X^i_{(\alpha)} X^k_{(\beta)} = -\delta_{(\alpha)(\beta)}, \quad g_{ik} X^i_{(\alpha)} U^k = 0 , \tag{68}$$

$$X^i_{(1)} X^k_{(1)} + X^i_{(2)} X^k_{(2)} + X^i_{(3)} X^k_{(3)} = -g^{ik} + U^i U^k = -\Delta^{ik} , \tag{69}$$

$$\epsilon^i{}_{klm} U^m X^k_{(\alpha)} X^l_{(\beta)} = \epsilon_{(\alpha)(\beta)(\gamma)} X^i_{(\gamma)}, \tag{70}$$

where $(\alpha), (\beta).... = (1), (2), (3)$ are tetrad indices; there is a summation over repeating indices. The permittivity tensor can be decomposed according to

$$\varepsilon^{ik} = -\sum_{(\alpha)} \varepsilon_{(\alpha)} X^i_{(\alpha)} X^k_{(\alpha)}, \quad \varepsilon^k_k = \varepsilon_{(1)} + \varepsilon_{(2)} + \varepsilon_{(3)}, \tag{71}$$

where the terms $\varepsilon_{(\alpha)}$ denote the eigenvalues, corresponding to the eigenvector $X^i_{(\alpha)}$. Since the tensor ε^{ik} is orthogonal to the four-velocity vector U^i, the corresponding eigenvalue $\varepsilon_{(0)}$ is equal to zero, and the velocity does not appear in this decomposition.

The structure of the B-tensor, $g^{im(B)}$, is much more sophisticated in the general case. Keeping in mind an application of this model to cosmology, we present here only one example, when the tetrad components of the impermeability tensor $(\mu^{-1})_{(\alpha)(\beta)}$ are diagonal, i.e.,

$$(\mu^{-1})_{(\alpha)(\beta)} = \frac{1}{\mu_{(\alpha)}}\delta_{(\alpha)(\beta)} \,, \tag{72}$$

and the tensor of cross-effects vanishes. In this case the B-tensor (see [46]) has the form

$$g^{im(B)} = g^{im(A)} \mp \hat{\mu}\sqrt{\frac{M_{(1)}M_{(2)}M_{(3)}}{\gamma}} \sum_{(\alpha)=(1)}^{(3)} \frac{\varepsilon_{(\alpha)}}{M_{(\alpha)}}X^i_{(\alpha)}X^m_{(\alpha)} \,, \tag{73}$$

where

$$M_{(\alpha)} \equiv 1 - \frac{\varepsilon_{(\alpha)}}{\mu_{(\alpha)}\hat{\mu}\varepsilon_{(1)}\varepsilon_{(2)}\varepsilon_{(3)}} \,. \tag{74}$$

2. Uniaxial case

Below we consider the applications of the model to the spacetimes with uniaxial three-dimensional subspaces and simplify the structure of the A- and B-tensors accordingly to this case. Let the direction of the privilege axis be Ox^3 and $M_{(1)} = M_{(2)} = 0$. Then one obtains

$$\varepsilon_\perp \equiv \varepsilon^1_1 = \varepsilon^2_2 = 1 + 2\mathcal{R}^{1t}{}_{1t} \,, \quad \varepsilon^3_3 \equiv \varepsilon_{||} = 1 + 2\mathcal{R}^{3t}{}_{3t} \,, \tag{75}$$

$$\frac{1}{\mu_\perp} \equiv (\mu^{-1})^1_1 = (\mu^{-1})^2_2 = 1 + 2\mathcal{R}^{23}{}_{23} \,, \quad (\mu^{-1})^3_3 \equiv \frac{1}{\mu_{||}} = 1 + 2\mathcal{R}^{12}{}_{12} \,, \tag{76}$$

$$\frac{1}{\hat{\mu}} = \varepsilon_{||}\varepsilon_\perp\mu_\perp \,, \quad \frac{1}{\gamma} = 1 - \frac{\varepsilon_{||}\mu_\perp}{\varepsilon_\perp\mu_{||}} \,. \tag{77}$$

The A- and B-tensors take the form

$$g^{ik(A)} = U^iU^k - \frac{1}{\varepsilon_{||}\mu_\perp}\left(X^i_{(1)}X^k_{(1)} + X^i_{(2)}X^k_{(2)}\right) - \frac{1}{\varepsilon_\perp\mu_\perp}X^i_{(3)}X^k_{(3)} \,, \tag{78}$$

$$g^{ik(B)} = U^iU^k - \frac{1}{\varepsilon_\perp\mu_{||}}\left(X^i_{(1)}X^k_{(1)} + X^i_{(2)}X^k_{(2)}\right) - \frac{1}{\varepsilon_\perp\mu_\perp}X^i_{(3)}X^k_{(3)} \,, \tag{79}$$

providing the relation (65). In other words, they give the associated metrics for the non-minimal EYMH model. In the WKB-approximation the Yang-Mills equations are satisfied when

$$g^{ik(A)}p_ip_k = 0 \,, \quad \text{or} \quad g^{ik(B)}p_ip_k = 0 \,, \tag{80}$$

(see Appendix B). Thus, these associated metrics can be interpreted as the color ones. This means, in particular, that the propagation of a test Yang-Mills wave in the non-minimally active spacetime can be described by two dispersion relations

$$\omega^2_{(A)} = \frac{p^2_\perp}{\varepsilon_{||}\mu_\perp} + \frac{p^2_{||}}{\varepsilon_\perp\mu_\perp} \,, \quad \omega^2_{(B)} = \frac{p^2_\perp}{\varepsilon_\perp\mu_{||}} + \frac{p^2_{||}}{\varepsilon_\perp\mu_\perp} \,, \tag{81}$$

where

$$\omega = p_lU^l \,, \quad p^2_\perp = -(p_1p^1 + p_2p^2) \,, \quad p^2_{||} = -p_3p^3 \,. \tag{82}$$

When the wave propagates in the longitudinal direction, i.e., $p_\perp = 0$, or in the transverse one, i.e., $p_{||} = 0$, the corresponding phase velocities can be defined as

$$\mathcal{V}_{||} \equiv \frac{\omega}{p_{||}}, \quad \mathcal{V}_\perp \equiv \frac{\omega}{p_\perp}. \tag{83}$$

For A-wave and B-wave they are, respectively,

$$\mathcal{V}_{||}^{(A)} = \mathcal{V}_{||}^{(B)} = \frac{1}{\sqrt{\varepsilon_\perp \mu_\perp}} = \sqrt{\frac{1 + 2\mathcal{R}^{23}{}_{23}}{1 + 2\mathcal{R}^{1t}{}_{1t}}},$$

$$\mathcal{V}_\perp^{(A)} = \frac{1}{\sqrt{\varepsilon_{||}\mu_\perp}} = \sqrt{\frac{1 + 2\mathcal{R}^{23}{}_{23}}{1 + 2\mathcal{R}^{3t}{}_{3t}}}, \quad \mathcal{V}_\perp^{(B)} = \frac{1}{\sqrt{\varepsilon_\perp \mu_{||}}} = \sqrt{\frac{1 + 2\mathcal{R}^{12}{}_{12}}{1 + 2\mathcal{R}^{1t}{}_{1t}}}. \tag{84}$$

These quantities depend on spacetime metric, and its derivatives and convert into one, when the non-minimal coupling constants vanish, i.e., $q_1 = q_2 = q_3 = 0$ (let us repeat, that in units used here the speed of light in the standard vacuum is equal to one). The color-acoustic metric can be represented in the form analogous to (78):

$$\tilde{g}^{ik} = \mathcal{A}^2 \left[U^i U^k - \frac{1}{\sigma_\perp^2} \left(X_{(1)}^i X_{(1)}^k + X_{(2)}^i X_{(2)}^k \right) - \frac{1}{\sigma_{||}^2} X_{(3)}^i X_{(3)}^k \right], \tag{85}$$

where

$$\mathcal{A}^2 = 1 + \mathcal{R}_t^t, \quad \sigma_\perp = \sqrt{\frac{1 + \mathcal{R}_t^t}{1 + \mathcal{R}_1^1}}, \quad \sigma_{||} = \sqrt{\frac{1 + \mathcal{R}_t^t}{1 + \mathcal{R}_3^3}}, \tag{86}$$

and $\sigma_\perp$ and $\sigma_{||}$ can be interpreted as phase velocities of a transversal and longitudinal waves, respectively.

V.   COSMOLOGICAL APPLICATION OF THE EFFECTIVE METRIC FORMALISM

A.   One-axis Bianchi-I model

Relativistic cosmology needs effective metric paradigm since it uses observational data obtained by analysis of the directional distribution and frequency properties of incoming photons. Propagating photons are influenced by numerous regular and stochastic phenomena, and we could try to identify this influence, if we reconstruct the effective dielectric permittivity and magnetic permeability of the regions, where the photons passed. Effective metric formalism gives mathematical grounds for such a reconstruction.

Let us use the metric

$$ds^2 = dt^2 - \left[a^2(t) \left(dx^2 + dy^2 \right) + c^2(t)dz^2 \right], \tag{87}$$

which is attributed to the well-known Bianchi-I cosmological model with two equivalent spatial directions, Ox and Oy. In [61] we presented the exact solutions of the non-minimally extended Einstein-Maxwell model with such metric in case when there is a magnetic field directed along $0z$. That model can be automatically generalized to the case of parallel Yang-Mills field $F_{ik}^{(a)} = q^{(a)} B(\delta_i^1 \delta_k^2 - \delta_i^2 \delta_k^1)$ along the line proposed in [60]. Thus, we can refer to the exact solutions, published in [61], as to the ones with non-minimally coupled parallel Yang-Mills field of the magnetic type. This means that we can use both terms: optical metric and color metric referring to that solutions. The tetrad vectors for this metric are

$$X_{(0)}^k = \delta_0^k, \quad X_{(1)}^k = \delta_1^k \frac{1}{a(t)}, \quad X_{(2)}^k = \delta_2^k \frac{1}{a(t)}, \quad X_{(3)}^k = \delta_3^k \frac{1}{c(t)}. \tag{88}$$

Based on the symmetry of the spacetime we easily obtain that the non-minimal susceptibility tensors $\mathcal{R}^{ik}{}_{mn}$ and $\Re_n^m$ correspond to the case of uniaxial symmetry. The non-vanishing components of these tensors are the following

$$-\mathcal{R}^{1t}{}_{1t} = \frac{1}{2}(4q_1 + 3q_2 + 2q_3)\frac{\ddot{a}}{a} + \frac{1}{2}(2q_1 + q_2)\frac{\ddot{c}}{c} + \frac{1}{2}(2q_1 + q_2)\frac{\dot{a}^2}{a^2} + \frac{1}{2}(4q_1 + q_2)\frac{\dot{a}}{a}\frac{\dot{c}}{c}, \tag{89}$$

$$-\mathcal{R}^{13}{}_{13}=\frac{1}{2}(4q_1+q_2)\frac{\ddot{a}}{a}+\frac{1}{2}(2q_1+q_2)\frac{\ddot{c}}{c}+\frac{1}{2}(2q_1+q_2)\frac{\dot{a}^2}{a^2}+\frac{1}{2}(4q_1+3q_2+2q_3)\frac{\dot{a}}{a}\frac{\dot{c}}{c}\,,\tag{90}$$

$$-\mathcal{R}^{3t}{}_{3t}=(2q_1+q_2)\frac{\ddot{a}}{a}+(q_1+q_2+q_3)\frac{\ddot{c}}{c}+q_1\frac{\dot{a}^2}{a^2}+(2q_1+q_2)\frac{\dot{a}}{a}\frac{\dot{c}}{c}\,,\tag{91}$$

$$-\mathcal{R}^{12}{}_{12}=(2q_1+q_2)\frac{\ddot{a}}{a}+q_1\frac{\ddot{c}}{c}+(q_1+q_2+q_3)\frac{\dot{a}^2}{a^2}+(2q_1+q_2)\frac{\dot{a}}{a}\frac{\dot{c}}{c}\,,\tag{92}$$

$$\mathcal{R}^{2t}{}_{2t}=\mathcal{R}^{1t}{}_{1t}\,,\quad \mathcal{R}^{23}{}_{23}=\mathcal{R}^{13}{}_{13}\,,\tag{93}$$

$$-\Re^t_t=(2q_4+q_5)\left(2\frac{\ddot{a}}{a}+\frac{\ddot{c}}{c}\right)+2q_4\left(\frac{\dot{a}^2}{a^2}+2\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right)\,,\tag{94}$$

$$-\Re^1_1=-\Re^2_2=(4q_4+q_5)\left(\frac{\ddot{a}}{a}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right)+2q_4\frac{\ddot{c}}{c}+(2q_4+q_5)\frac{\dot{a}^2}{a^2}\,,\tag{95}$$

$$-\Re^3_3=2q_4\left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)+(2q_4+q_5)\left(\frac{\ddot{c}}{c}+2\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right)\,.\tag{96}$$

Thus, we are ready to write the effective metrics in terms of permittivity (75) and impermeability (76).

1. Color (optical) metrics

The A-metric and B-metric are, respectively,

$$g^{ik(A)}=\delta^i_t\delta^k_t-\frac{1}{\varepsilon_{||}\mu_\perp a^2}\left(\delta^i_1\delta^k_1+\delta^i_2\delta^k_2\right)-\frac{1}{\varepsilon_\perp\mu_\perp c^2}\delta^i_3\delta^k_3\,,\tag{97}$$

$$g^{ik(B)}=\delta^i_t\delta^k_t-\frac{1}{\varepsilon_\perp\mu_{||}a^2}\left(\delta^i_1\delta^k_1+\delta^i_2\delta^k_2\right)-\frac{1}{\varepsilon_\perp\mu_\perp c^2}\delta^i_3\delta^k_3\,,\tag{98}$$

where $\varepsilon_{||}$, $\varepsilon_\perp$, $\mu_{||}$ and $\mu_\perp$ are given by (75) and (76) with the susceptibility tensors (89)-(96). The relative anisotropy of the permittivity/impermeability can be estimated by the following quantities

$$\varepsilon_\perp-\varepsilon_{||}=(q_2+2q_3)\left(\frac{\ddot{c}}{c}-\frac{\ddot{a}}{a}\right)+q_2\,\frac{\dot{a}}{a}\left(\frac{\dot{c}}{c}-\frac{\dot{a}}{a}\right)\,,\tag{99}$$

$$\frac{1}{\mu_\perp}-\frac{1}{\mu_{||}}=q_2\left(\frac{\ddot{c}}{c}-\frac{\ddot{a}}{a}\right)+(2q_3+q_2)\frac{\dot{a}}{a}\left(\frac{\dot{c}}{c}-\frac{\dot{a}}{a}\right)\,.\tag{100}$$

It is worth noting that q_1 does not enter the formulas (99) and (100). The isotropization of the Universe leads to the vanishing of these differences.

Special case $q_2+q_3=0$
It is clear from (89)-(93) that in this special case four non-vanishing components of the non-minimal susceptibility coincide:

$$\mathcal{R}^{2t}{}_{2t}=\mathcal{R}^{1t}{}_{1t}=\mathcal{R}^{23}{}_{23}=\mathcal{R}^{13}{}_{13}\,,\tag{101}$$

and, thus, the permittivity scalars are linked by

$$\varepsilon_\perp\mu_\perp=1\,,\quad \varepsilon_{||}\mu_{||}=1\,,\tag{102}$$

which provide the relations

$$\mathcal{V}_{||}^{(A)} = \mathcal{V}_{||}^{(B)} = 1, \quad \mathcal{V}_{\perp}^{(A)}\mathcal{V}_{\perp}^{(B)} = 1. \tag{103}$$

The color metrics in this case can also be rewritten as the functions of only one parameter, say, $\nu^2 = \frac{\varepsilon_{||}}{\varepsilon_\perp}$:

$$g^{ik(A)} = U^i U^k + \frac{1}{\nu^2}\Delta^{ik} + \left(\frac{1}{\nu^2} - 1\right) X_{(3)}^i X_{(3)}^k, \tag{104}$$

$$g^{ik(B)} = U^i U^k + \nu^2 \Delta^{ik} + \left(\nu^2 - 1\right) X_{(3)}^i X_{(3)}^k. \tag{105}$$

The structure of these color (optical) metrics shows that the waves propagate in the longitudinal direction with the speed of light in standard vacuum. As for the waves propagating in the orthogonal direction, the phase velocity of one of them is less than speed of light in vacuum, the second wave being superluminal. We deal with birefringence.

2. Color-acoustic metric

Using (94)-(96) one can write the color-acoustic metric in the form

$$\tilde{g}^{ik} = \mathcal{A}^2(t)\left\{\delta_t^i\delta_t^k - \frac{1}{a^2(t)\sigma_\perp^2(t)}\left(\delta_1^i\delta_1^k + \delta_2^i\delta_2^k\right) - \frac{1}{c^2(t)\sigma_{||}^2(t)}\delta_3^i\delta_3^k\right\}, \tag{106}$$

where

$$\mathcal{A}^2(t) \equiv 1 - (2q_4 + q_5)\left(2\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c}\right) - 2q_4\left[\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right], \tag{107}$$

$$\sigma_{||}^2(t) \equiv \frac{1 - (2q_4 + q_5)\left(2\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c}\right) - 2q_4\left[\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right]}{1 - 2q_4\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] - (2q_4 + q_5)\left(\frac{\ddot{c}}{c} + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right)}, \tag{108}$$

$$\sigma_\perp^2(t) \equiv \frac{1 - (2q_4 + q_5)\left(2\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c}\right) - 2q_4\left[\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right]}{1 - (4q_4 + q_5)\left(\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\frac{\dot{c}}{c}\right) - 2q_4\frac{\ddot{c}}{c} - (2q_4 + q_5)\left(\frac{\dot{a}}{a}\right)^2}. \tag{109}$$

The eikonal equation for the scalar fields reads

$$\tilde{g}^{ik}P_i P_k = m^2. \tag{110}$$

Here the following definitions are introduced:

$$\mathcal{E}^2 = m^2\mathcal{A}^{-2} + P_{||}^2\sigma_{||}^{-2} + P_\perp^2\sigma_\perp^{-2}, \tag{111}$$

$$\mathcal{E}^2 \equiv P_t P^t, \quad P_\perp^2 \equiv -(P_1 P^1 + P_2 P^2), \quad P_{||}^2 \equiv -P_3 P^3, \tag{112}$$

where P_i is particle four-momentum, $\mathcal{E}$ is its energy. When a scalar particle moves along longitudinal direction, i.e., $P_\perp = 0$, its three-velocity is

$$v_{||}(t) = \frac{P_{||}}{\mathcal{E}} = \sigma_{||}\sqrt{1 - m^2\mathcal{A}^{-2}\mathcal{E}^{-2}}, \tag{113}$$

thus, for high-energy scalar particle $(\mathcal{E} \gg m)$ the quantity $\sigma_{||}$ gives asymptotic longitudinal velocity. When $P_{||} = 0$, one obtains, respectively,

$$v_\perp(t) = \frac{P_\perp}{\mathcal{E}} = \sigma_\perp\sqrt{1 - m^2\mathcal{A}^{-2}\mathcal{E}^{-2}}, \tag{114}$$

thus, $\sigma_\perp$ is some asymptotic transversal velocity. The ratio

$$\frac{v_{||}(t)}{v_\perp(t)} = \frac{\sigma_{||}(t)}{\sigma_\perp(t)} = \sqrt{\frac{1 - (4q_4 + q_5)\left(\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\frac{\dot{c}}{c}\right) - 2q_4\frac{\ddot{c}}{c} - (2q_4 + q_5)\left(\frac{\dot{a}}{a}\right)^2}{1 - 2q_4\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] - (2q_4 + q_5)\left(\frac{\ddot{c}}{c} + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right)}} \tag{115}$$

does not depend on the particle energy $\mathcal{E}$ and is predetermined by the values of the functions $a(t)$ and $c(t)$ and their derivatives, as well as by the values of the non-minimal coupling parameters q_4 and q_5. Note, that when $q_5 = 0$ and the derivative coupling is absent, this ratio is equal to one.

3. Exactly integrable example

Let us extract from [61] one of the exact solutions of the non-minimally extended Bianchi-I model with magnetic field ($\mathbf{\Phi} = 0$). This solution is characterized by the following features. The magnetic field is $B(t) = B(t_0)\frac{a^2(t_0)}{a^2(t)}$. It can be replaced in our model by the "parallel" Yang-Mills field of the magnetic type with $F_{12}^{(a)} = B(t_0)a^2(t_0)\delta_{(3)}^{(a)}$. The longitudinal $\mathcal{P}_{||}$ and transverse $\mathcal{P}_\perp$ pressures of the matter are connected with the energy density W as follows: $\mathcal{P}_{||} = -\mathcal{P}_\perp = -W$. The solution for the $a(t)$ is of de Sitter type

$$a(t) = a(t_0)e^{H(t-t_0)}, \quad \Lambda = 3H^2, \tag{116}$$

with the constant H given by

$$H^2 = \frac{2W(t_0) + B^2(t_0)}{10q_1 B^2(t_0)}, \tag{117}$$

the coupling constants being linked by the relation $6q_1 + 4q_2 + q_3 = 0$. The solution for $c(t)$ is

$$c(t) = c(t_0)e^{H(t-t_0)}\frac{\left[1 - \alpha e^{-4H(t-t_0)}\right]}{(1-\alpha)}, \quad \alpha \equiv \kappa q_1 B^2(t_0). \tag{118}$$

Clearly, this non-minimal cosmological model is non-singular for $t \geq t_0$, if $\alpha < 1$, i.e., the first coupling parameter q_1 satisfies inequality $0 < q_1 < \frac{1}{\kappa B^2(t_0)}$. For such model the relative anisotropy can be characterized by

$$\varepsilon_\perp - \varepsilon_{||} = \frac{2\kappa(q_2 + 4q_3)[2W(t_0) + B^2(t_0)]}{5\left[\alpha - e^{4H(t-t_0)}\right]}, \tag{119}$$

$$\frac{1}{\mu_{||}} - \frac{1}{\mu_\perp} = \frac{2\kappa(q_2 - 2q_3)[2W(t_0) + B^2(t_0)]}{5\left[\alpha - e^{4H(t-t_0)}\right]}. \tag{120}$$

The anisotropy of the permittivity disappears exponentially at $t \to \infty$, i.e., when Bianchi-I model isotropizes. Mention that $\varepsilon_\perp = \varepsilon_{||}$, when $q_2 + 4q_3 = 0$, as well as, $\frac{1}{\mu_{||}} = \frac{1}{\mu_\perp}$, when $q_2 = 2q_3$, nevertheless, it can not occur simultaneously, since when $q_2 = q_3 = 0$, q_1 also vanishes and the non-minimal model degenerates. Analogously, the color-acoustic anisotropy is characterized by the relation

$$\left(\frac{\sigma_{||}}{\sigma_\perp}\right)^2 = \frac{1 - \alpha e^{-4H(t-t_0)} - H^2\left[3(4q_4 + q_5) + (q_5 - 12q_4)\alpha e^{-4H(t-t_0)}\right]}{[1 - 3H^2(4q_4 + q_5)]\left[1 - \alpha e^{-4H(t-t_0)}\right]}. \tag{121}$$

Of course, at $t \to \infty$, this quantity tends to one, $\frac{\sigma_{||}}{\sigma_\perp} \to 1$, as it should be.

The analysis of the expressions (84) for the given model shows, that for generic q_2 and q_3 there are, in principle, two critical moments of time, when the phase velocities vanish and the A-wave or/and B-wave stop. At the first moment, t_1^*, the phase velocities of both waves propagating in the longitudinal direction, as well as of the A-wave moving in the orthogonal direction, vanish. It is possible, when

$$1 + 2\mathcal{R}^{23}{}_{23}(t_1^*) = 0 \quad \rightarrow \quad t_1^* = t_0 + \frac{1}{4H}\ln\left\{\frac{\alpha[1 + 2H^2(3q_2 + 4q_3)]}{1 + 2H^2 q_2}\right\}, \tag{122}$$

if the argument of logarithm is more than one. At the second critical moment, t_2^*, the B-wave propagating in the transverse directions has vanishing phase velocity, it is possible, when

$$1 + 2\mathcal{R}^{12}{}_{12}(t_2^*) = 0 \quad \rightarrow \quad t_2^* = t_0 + \frac{1}{4H} \ln \left\{ \frac{\alpha[1 + 10H^2 q_2)]}{1 + 2H^2 q_2} \right\}, \tag{123}$$

if the argument of this logarithm is also more than one. In its turn, we can choose the parameters α, q_2 and q_3 so that both optical metrics are regular at each time. Let us illustrate this possibility by the example, when $q_2 = -2q_1 < 0$ and $q_3 = 2q_1 > 0$, satisfying the basic condition $6q_1 + 4q_2 + q_3 = 0$. For this model the waves moving in the longitudinal direction have the phase velocity equal to one, as for the waves propagating in the transverse direction, they are characterized by the following reduced formulas

$$\left(\mathcal{V}_\perp^{(A)} \right)^2 = \frac{(1 - 4q_1 H^2) - \alpha(1 + 4q_1 H^2)e^{-4H(t-t_0)}}{(1 - 4q_1 H^2) - \alpha(1 - 20q_1 H^2)e^{-4H(t-t_0)}}, \quad \mathcal{V}_\perp^{(B)} \mathcal{V}_\perp^{(A)} = 1. \tag{124}$$

Taking (117) into account, we obtain that the square of the phase velocity (124) is positive for $t > t_0$, when

$$\frac{2}{5} < \alpha < \frac{3}{7}, \quad \frac{4W(t_0)}{B^2(t_0)} < \min \left\{ \frac{5\alpha - 2}{6 - 5\alpha}, \frac{3 - 7\alpha}{1 + \alpha} \right\}. \tag{125}$$

Analogously, the longitudinal and transversal color-acoustic waves stop, when, respectively,

$$t_3^* = t_0 + \frac{1}{4H} \ln \left\{ \frac{\alpha[1 + H^2(q_5 - 12q_4)]}{1 - 3H^2(4q_4 + q_5)} \right\}, \tag{126}$$

$$t_4^* = t_0 + \frac{1}{4H} \ln \alpha. \tag{127}$$

The color-acoustic metric degenerates when $\mathcal{A}(t_5^*) = 0$, where

$$t_5^* = t_0 + \frac{1}{4H} \ln \left\{ \frac{\alpha[1 - H^2(12q_4 + 11q_5)]}{1 - 3H^2(4q_4 + q_5)} \right\}. \tag{128}$$

Let us mention that the color-acoustic metrics does not exist initially, if the non-minimal parameters are coupled by

$$10q_1 = (4q_4 + q_5)[2W(t_0) + B^2(t_0)]. \tag{129}$$

To avoid the singularities in the color-acoustic metric one can put, e.g., $q_5 = -4q_4 < 0$, providing the expressions (107) and (121) to be positive for $t > t_0$.

B. Spatially flat FLRW model

FLRW model with $k = 0$ (i.e., spatially flat model) can be obtained from the previous model, if we put $c(t) = a(t)$. The formulas for the A- and B- metrics can be converted into

$$g^{ik(A)} = g^{ik(B)} = \delta_t^i \delta_t^k - \frac{1}{\varepsilon \mu a^2} \left(\delta_1^i \delta_1^k + \delta_2^i \delta_2^k + \delta_3^i \delta_3^k \right), \tag{130}$$

where

$$\varepsilon \equiv 1 - 2(3q_1 + 2q_2 + q_3)\frac{\ddot{a}}{a} - 2(3q_1 + q_2) \left(\frac{\dot{a}}{a} \right)^2, \tag{131}$$

$$\mu \equiv 1 - 2(3q_1 + q_2)\frac{\ddot{a}}{a} - 2(3q_1 + 2q_2 + q_3) \left(\frac{\dot{a}}{a} \right)^2. \tag{132}$$

Effective refractive index $n_{(\text{eff})}$ is given by

$$n_{(\text{eff})}^2 \equiv \varepsilon \mu a^2 = a^2 \frac{[a^2 - 2(3q_1 + 2q_2 + q_3)\ddot{a}a - 2(3q_1 + q_2)\dot{a}^2]}{[a^2 - 2(3q_1 + q_2)\ddot{a}a - 2(3q_1 + 2q_2 + q_3)\dot{a}^2]}. \tag{133}$$

The effective refractive index coincides with the standard one, i.e., $n_{(\text{eff})} = a(t)$, first, when $q_2 + q_3 = 0$, second, when $\frac{\dot{a}}{a} = H_0 = \text{const}$. When the law of isotropic expansion is the power-like one, say, $a(t) = a(t_0)\left(\frac{t}{t_0}\right)^\omega$, then (133) reduces to

$$\frac{n^2_{(\text{eff})}}{a^2(t)} = \frac{t^2 - 2\omega\left[\omega(6q_1 + 3q_2 + q_3) - (3q_1 + 2q_2 + q_3)\right]}{t^2 - 2\omega\left[\omega(6q_1 + 3q_2 + q_3) - (3q_1 + q_2)\right]}. \tag{134}$$

At the moment t^*

$$t^* = \sqrt{2\omega\left[\omega(6q_1 + 3q_2 + q_3) - (3q_1 + q_2)\right]} \tag{135}$$

the color (optical) wave stops, since the phase velocity $V_{ph} = \frac{a}{n_{(\text{eff})}}$ vanishes, of course, the expression in the square root has to be positive. There is a number of sub-models in which the singularities do not appear. For instance, when $q_1 = -q$, $q_2 = 2q$, $q_3 = -q$, $q > 0$ (this model has been considered in [23, 25] for the spherically symmetric static case) and $\omega > 1$, the expression (134) is positive for each time. Color-acoustic metric reads

$$\tilde{g}^{ik} = \tilde{\mathcal{A}}^2(t)\left[\delta_t^i\delta_t^k - \frac{1}{a^2(t)\sigma^2(t)}\left(\delta_1^i\delta_1^k + \delta_2^i\delta_2^k + \delta_3^i\delta_3^k\right)\right]. \tag{136}$$

$$\tilde{\mathcal{A}}^2(t) \equiv 1 - 3(2q_4 + q_5)\frac{\ddot{a}}{a} - 6q_4\left(\frac{\dot{a}}{a}\right)^2, \tag{137}$$

$$\sigma^2(t) = \sigma_\perp^2(t) = \sigma_\parallel^2(t) = \frac{1 - 3(2q_4 + q_5)\frac{\ddot{a}}{a} - 6q_4\left(\frac{\dot{a}}{a}\right)^2}{1 - (6q_4 + q_5)\frac{\ddot{a}}{a} - (6q_4 + 2q_5)\left(\frac{\dot{a}}{a}\right)^2}, \tag{138}$$

The quantity σ differs from one if and only if $q_5 \neq 0$, i.e., the derivative coupling of the Higgs fields with curvature is present. Finally, this color-acoustic metric is non-singular, if, for instance, $q_5 = -2q_4 > 0$ and $\omega > \frac{2}{3}$.

C. De Sitter model

We consider now the model with positive curvature, $K > 0$, and reduce the metric to the de Sitter form [62]

$$ds^2 = dt^2 - \exp\{2\sqrt{K}t\}(\delta_{\alpha\beta}dx^\alpha dx^\beta). \tag{139}$$

For this spacetime the Riemann, Ricci tensors and Ricci scalar, respectively, take the form

$$R_{ikmn} = -K\left(g_{im}g_{kn} - g_{in}g_{km}\right), \quad R_{ik} = -3Kg_{ik}, \quad R = -12K. \tag{140}$$

The tensors $\mathcal{R}_{ikmn}$ (16) and $\Re_{ik}$ (17) yield

$$\mathcal{R}_{ikmn} = -K(6q_1 + 3q_2 + q_3)\left(g_{im}g_{kn} - g_{in}g_{km}\right), \quad \Re_{ik} = -3K(4q_4 + q_5)g_{ik}. \tag{141}$$

Let us mention that, when we deal with de Sitter model, only two combinations of the coupling parameters $\lambda_1 = 6q_1 + 3q_2 + q_3$ and $\lambda_4 = q_4 + \frac{1}{4}\lambda_5$ (see (24)) enter the susceptibility tensors $\mathcal{R}_{ikmn}$ and $\Re_{ik}$. In this subsection we use the parameters λ_1 and λ_4 instead of $q_1,...q_5$ in order to shorten the formulas. Then, the $\mathbf{H}^{ik}$ tensor and the $\mathbf{\Psi}^m$ vector simplify significantly, and the equations (28) and (30) convert, respectively, into

$$(1 - 2\lambda_1 K)\,\hat{D}_k\mathbf{F}^{ik} = \mathcal{G}^2\left(1 - 12\lambda_4 K\right)[\hat{D}^i\mathbf{\Phi}, \mathbf{\Phi}], \tag{142}$$

$$(1 - 12\lambda_4 K)\,\hat{D}_k\hat{D}^k\mathbf{\Phi} = -m^2\mathbf{\Phi}. \tag{143}$$

Thus, when $\lambda_1 < \frac{1}{2K}$ and $\lambda_4 < \frac{1}{12K}$ the color and color-acoustic metrics are simply proportional to the spacetime one

$$g^{(A)ik} = g^{(B)ik} = g^{*ik} = \sqrt{1 - 2\lambda_1 K}\,g^{ik}, \tag{144}$$

$$\tilde{g}^{(A)ik} = \tilde{g}^{(B)ik} = \sqrt{1 - 12\lambda_4 K} \ g^{ik} \,. \tag{145}$$

One can redefine the coupling constant $\mathcal{G}$ and the mass m

$$\mathcal{G} \to \mathcal{G}^* = \mathcal{G}\sqrt{\frac{1-12\lambda_4 K}{1-2\lambda_1 K}}, \quad m \to m^* = m\sqrt{\frac{1}{1-12\lambda_4 K}}, \tag{146}$$

so that the Yang-Mills and Higgs equations take the standard (minimal) form, but the constants are non-minimally redefined. In the WKB-approximation the conditions $g^{*ik}p_i p_k = 0$ and $g^{ik}p_i p_k = 0$ are in fact equivalent, thus, the massless test particles propagate in the "non-minimally active" de Sitter model as in standard de Sitter one. This fact has been emphasized above during the analysis of the FLRW model. The special case, when $2K(6q_1+3q_2+q_3)=1=2\lambda_1 K$ and $3K(4q_4+q_5)=1=12\lambda_4 K$, related to the so-called hidden Yang-Mills and Higgs fields, is discussed in [21].

VI. CONCLUSIONS

1. We developed the concept of effective metrics and introduced the associated, color and color-acoustic metrics, attributed to the Einstein-Yang-Mills-Higgs model. The procedure of reconstruction of the effective metrics consists of two steps. The first one, the reconstruction of the associated metrics, is based on an original multi-metric representations (54) and (61) of the constitutive tensors for the Yang-Mills and Higgs fields, respectively. The second step, the identification of the associated metrics with color and color-acoustic ones, is provided by the WKB-analysis of the master equations and by the well-known analogy with optical metrics.
2. Curvature interactions between gravitational, gauge and scalar fields are described in terms of five-parameter non-minimal EYMH model, and the basic ideas concerning associated, color and color-acoustic metrics are realized. The color (see (78), (79) with (75), (76)) and the color-acoustic (85) metrics for the uniaxial models are reconstructed explicitly.
3. The formalism of effective metrics is applied to Bianchi-I, FLRW and de Sitter cosmological models, and the associated, color and color-acoustic metrics are represented explicitly in terms of metric coefficients of the corresponding spacetimes and their derivatives. Phase velocities of the corresponding non-minimally coupled color (84) and color-acoustic (86) waves are the main distinguishing features in these expressions. These phase velocities and their ratios are calculated for all models (see, e.g., (108), (109), (115), (121), (124)). Thus, we obtained the tool for a qualitative analysis of the problem of propagation of test color, optical and scalar waves in the "non-minimally active" vacuum, interacting with curvature.
4. The analysis of the presented models allows us to conclude that for generic set of coupling constants $q_1, q_2, \ldots q_5$ the color and color-acoustic metrics are singular. This means that generally the phase velocities of the corresponding waves can be equal to zero or infinity at specific time moments (see, e.g., (122), (123), (126), (135)). The first case relates to the stopping of the corresponding wave. The second one is associated with the appearance of the analogs of trapped surfaces, apparent horizons or event horizons, which are discussed in detail in the literature devoted to analog gravity [36, 37, 38].
5. In its turn we present the examples of the sets of coupling constants $q_1, q_2, \ldots q_5$, for which there are no singularities in color and color-acoustic metrics. Such (regular) models predict, in particular, that the phase velocities of the waves essentially depend on the time moment, on the direction of propagation and are very sensitive to the choice of the values of the free parameters of the model. These conclusions are in agreement with the ones made in numerous papers, devoted to the variation of the speed of light under the influence of curvature interactions in electrodynamics (see, e.g., [63] - [69] and references therein). These conclusions require to attract a special attention to the analysis of the birefringence and time delay effects in observational astronomy, since the photons registered simultaneously are not necessarily emitted at the same time. We hope to devote a special paper to the study of such problems.

APPENDIX A: SYMMETRY OF THE CONSTITUTIVE TENSORS

1. Tensor $C_{(a)(b)}^{ikmn}$

Model under discussion (see (15) and (16)) is characterized by the tensor $C_{(a)(b)}^{ikmn}$ symmetric with respect to indices (a) and (b), as well as symmetric with respect to transposition of the pairs of indices ik and mn. In the general case this symmetry can be broken, and the tensor $C_{(a)(b)}^{ikmn}$ can loose the symmetry properties $C_{(a)(b)}^{ikmn} = C_{(b)(a)}^{ikmn}$ and

$C_{(a)(b)}^{ikmn} = C_{(a)(b)}^{mnik}$ separately. For $U(1)$ gauge invariant electrodynamics the antisymmetric part of C^{ikmn}:

$$C_{(\text{skewon})}^{ikmn} \equiv \frac{1}{2}(C^{ikmn} - C^{mnik}), \tag{A1}$$

introduced by Hehl and Obukhov [42] and indicated as skewon, disappears from the Lagrangian. In the $SU(n)$ symmetric Yang-Mills-Higgs theory $(n > 1)$ the skewonic terms can appear in the Lagrangian explicitly, since the strength field tensor $F_{ik}^{(a)}$ has a group index with two and more values, and indices (a) and (b) can be in the antisymmetric combination. Generally, we have the following decomposition of the second term in the Lagrangian (1):

$$\frac{1}{2}C_{(a)(b)}^{ikmn}F_{ik}^{(a)}F_{mn}^{(b)} = \frac{1}{2}\Theta_{(a)(b)}^{ikmn}F_{ik}^{(a)}F_{mn}^{(b)} + \frac{1}{2}\Xi_{(a)(b)}^{ikmn}F_{ik}^{(a)}F_{mn}^{(b)}, \tag{A2}$$

where the tensors $\Theta_{(a)(b)}^{ikmn}$ and $\Xi_{(a)(b)}^{ikmn}$ possess the following symmetry of indices

$$\Theta_{(a)(b)}^{ikmn} = \Theta_{(a)(b)}^{mnik} = \Theta_{(b)(a)}^{ikmn}, \quad \Xi_{(a)(b)}^{ikmn} = -\Xi_{(a)(b)}^{mnik} = \Xi_{(b)(a)}^{mnik}. \tag{A3}$$

The term $\Xi_{(a)(b)}^{ikmn}$ explicitly represents the Yang-Mills skewon, if we use the terminology introduced by Hehl and Obukhov [42]. In this paper we do not consider such a model and assume that $\Xi_{(a)(b)}^{ikmn} = 0$.

The symmetric part of the tensor of linear response $\Theta_{(a)(b)}^{ikmn}$ can be decomposed using the velocity four-vector U^i in the same way as in electrodynamics:

$$\Theta_{(a)(b)}^{ikmn} = \frac{1}{2}\left(\varepsilon_{(a)(b)}^{im}U^kU^n - \varepsilon_{(a)(b)}^{in}U^kU^m + \varepsilon_{(a)(b)}^{kn}U^iU^m - \varepsilon_{(a)(b)}^{km}U^iU^n\right)$$

$$+ \frac{1}{2}\left[-\eta^{ik}{}_l(\mu^{-1})_{(a)(b)}^{ls}\eta^{mn}{}_s + \eta^{ik}{}_l(U^m\nu_{(a)(b)}^{ln} - U^n\nu_{(a)(b)}^{lm}) + \eta_l{}^{mn}(U^i\nu_{(a)(b)}^{lk} - U^k\nu_{(a)(b)}^{li})\right]. \tag{A4}$$

Here $\varepsilon_{(a)(b)}^{im}$ and $(\mu^{-1})_{(a)(b)}^{pq}$ are analogs of the tensors of dielectric permittivity and magnetic impermeability, respectively, and $\nu_{(a)(b)}^{pm}$ is an analog of the tensor of magneto-electric coefficients. These quantities are defined as

$$\varepsilon_{(a)(b)}^{im} = 2\Theta_{(a)(b)}^{ikmn}U_kU_n, \quad (\mu^{-1})_{(a)(b)}^{pq} = -\frac{1}{2}\eta^p{}_{ik}\Theta_{(a)(b)}^{ikmn}\eta_{mn}{}^q, \tag{A5}$$

$$\nu_{(a)(b)}^{pm} = \eta^p{}_{ik}\Theta_{(a)(b)}^{ikmn}U_n = U_k\Theta_{(a)(b)}^{mkln}\eta_{ln}{}^p. \tag{A6}$$

Tensors η_{mnl} and η^{ikl} are the antisymmetric tensors orthogonal to U^i defined as

$$\eta_{mnl} \equiv \epsilon_{mnls}U^s, \quad \eta^{ikl} \equiv \epsilon^{ikls}U_s. \tag{A7}$$

They are connected by the useful identities

$$-\eta^{ikp}\eta_{mnp} = \delta_{mns}^{ikl}U_lU^s = \Delta_m^i\Delta_n^k - \Delta_n^i\Delta_m^k, \quad \frac{1}{2}\eta^{ikl}\eta_{klm} = -\delta_{ms}^{il}U_lU^s = -\Delta_m^i, \tag{A8}$$

where δ_{mns}^{ikl} and δ_{ms}^{il} are the generalized Kronecker tensors [59]. The symmetric projection tensor Δ^{ik} is defined as $\Delta^{ik} = g^{ik} - U^iU^k$. The tensors $\varepsilon_{(a)(b)}^{ik}$ and $(\mu^{-1})_{(a)(b)}^{ik}$ are symmetric, but $\nu_{(a)(b)}^{lk}$ is generally non-symmetric with respect to the spacetime indices. These three tensors are orthogonal to U^i,

$$\varepsilon_{(a)(b)}^{ik}U_k = 0, \quad (\mu^{-1})_{(a)(b)}^{ik}U_k = 0, \quad \nu_{(a)(b)}^{lk}U_l = 0 = \nu_{(a)(b)}^{lk}U_k, \tag{A9}$$

and possess 21 independent components for any fixed set of the indices (a) and (b).

To complete the analogy, we introduce two four-vectors of excitation $\mathcal{D}_{(a)}^i$ and $\mathcal{H}_{(a)}^i$, as well as, two four-vectors of the field strength $\mathcal{E}_{(a)}^i$ and $\mathcal{B}_{(a)}^i$ by the definitions

$$\mathcal{D}_{(a)}^i = \mathcal{H}_{(a)}^{ik}U_k, \quad \mathcal{H}_{(a)}^i = \mathcal{H}_{(a)}^{*ik}U_k, \quad \mathcal{E}_i^{(a)} = F_{ik}^{(a)}U^k, \quad \mathcal{B}_i^{(a)} = F_{ik}^{*(a)}U^k, \tag{A10}$$

where the four-vector of the velocity U^i is normalized by unity: $U^i U_i = 1$. These vectors are orthogonal to the velocity four-vector U^i:

$$\mathcal{D}^i_{(a)} U_i = 0 = \mathcal{E}^{(a)}_i U^i , \quad \mathcal{H}^i_{(a)} U_i = 0 = \mathcal{B}^{(a)}_i U^i , \tag{A11}$$

and form the basis for the decomposition of the $F^{(a)}_{mn}$ and $\mathcal{H}^{mn}_{(a)}$ tensors:

$$F^{(a)}_{mn} = \mathcal{E}^{(a)}_m U_n - \mathcal{E}^{(a)}_n U_m - \epsilon_{mn\cdot\cdot}{}^{ls} U_s \mathcal{B}^{(a)}_l , \quad \mathcal{H}^{mn}_{(a)} = \mathcal{D}^m_{(a)} U^n - \mathcal{D}^n_{(a)} U^m - \epsilon^{mn}_{\cdot\cdot\ ls} U^s \mathcal{H}^l_{(a)} . \tag{A12}$$

When we deal with the five-parameter EYMH model, the direct calculations using (26) show that

$$\varepsilon^{im}_{(a)(b)} = \left[\Delta^{im} + 2\mathcal{R}^{ikmn} U_k U_n \right] G_{(a)(b)} , \tag{A13}$$

$$(\mu^{-1})^{pq}_{(a)(b)} = G_{(a)(b)} \left[\Delta^{pq} - 2 \, {}^*\mathcal{R}^{*plqs} U_l U_s \right] , \tag{A14}$$

$$\nu^{pm}_{(a)(b)} = -{}^*\mathcal{R}^{plnm} U_l U_n G_{(a)(b)} . \tag{A15}$$

Thus, one can state that the non-minimal coupling of the gravitational and Yang-Mills-Higgs field effectively changes the properties of the vacuum linear response. Particularly, the tensor of non-minimal susceptibility $\mathcal{R}^{ijkl}$ determines the variation of color electric-type permittivity, the corresponding double dual tensor ${}^*\mathcal{R}^{*plqs}$ is responsible for the changes in the color magnetic-type impermeability tensor, and non-vanishing ${}^*\mathcal{R}^{plqs}$ produces the so-called color magneto-electric-type properties of the non-minimal vacuum.

2.   Tensor $\mathcal{C}^{ik}_{(a)(b)}$

The third term in the Lagrangian (1) also contains two parts

$$-\mathcal{C}^{ik}_{(a)(b)} \hat{D}_i \Phi^{(a)} \hat{D}_k \Phi^{(b)} = -\Omega^{ik}_{(a)(b)} \hat{D}_i \Phi^{(a)} \hat{D}_k \Phi^{(b)} - \Psi^{ik}_{(a)(b)} \hat{D}_i \Phi^{(a)} \hat{D}_k \Phi^{(b)} , \tag{A16}$$

where tensors $\Omega^{ik}_{(a)(b)}$ and $\Psi^{ik}_{(a)(b)}$ are symmetric and antisymmetric, respectively, i.e.,

$$\Omega^{ik}_{(a)(b)} = \Omega^{ki}_{(a)(b)} = \Omega^{ik}_{(b)(a)} , \quad \Psi^{ik}_{(a)(b)} = -\Psi^{ki}_{(a)(b)} = \Psi^{ki}_{(b)(a)} . \tag{A17}$$

The decomposition of the symmetric tensor $\Omega^{ik}_{(a)(b)}$ has the same structure as a symmetric stress-energy tensor

$$\Omega^{ik}_{(a)(b)} = \mathcal{A}_{(a)(b)} U^i U^k + \mathcal{J}^i_{(a)(b)} U^k + \mathcal{J}^k_{(a)(b)} U^i + \mathcal{B}^{ik}_{(a)(b)} , \tag{A18}$$

where $\mathcal{J}^i_{(a)(b)}$ and $\mathcal{B}^{ik}_{(a)(b)}$ are orthogonal to the velocity four-vector U^k. For the non-minimal vacuum we obtain

$$\mathcal{A}_{(a)(b)} = G_{(a)(b)} \left(1 + q_4 R + q_5 R^{ik} U_i U_k \right) , \tag{A19}$$

$$\mathcal{J}^i_{(a)(b)} = G_{(a)(b)} q_5 \left(R^{im} - U^i R^{mn} U_n \right) U_m , \tag{A20}$$

$$\mathcal{B}^{ik}_{(a)(b)} = G_{(a)(b)} \left[(1 + q_4 R) \Delta^{ik} + q_5 R^{mn} \Delta^i_m \Delta^k_n \right] . \tag{A21}$$

The skew-symmetric part $\Psi^{ik}_{(a)(b)}$ has the decomposition analogous to the Maxwell tensor

$$\Psi^{ik}_{(a)(b)} = \mathcal{M}^i_{(a)(b)} U^k - \mathcal{M}^k_{(a)(b)} U^i - \eta^{ikl} \Lambda_{l(a)(b)} , \tag{A22}$$

with

$$\mathcal{M}^i_{(a)(b)} = -\mathcal{M}^i_{(b)(a)} , \quad \Lambda_{l(a)(b)} = -\Lambda_{l(b)(a)} , \tag{A23}$$

and introduces the scalar skewons by the analogy with the terminology of Hehl and Obukhov.

3. Tensor $\mathcal{C}_{ijkl}{}^{mnpq}$

This tensor can also be decomposed into irreducible parts using the same procedure as in case of $C^{ijkl}_{(a)(b)}$, nevertheless, we do not present this procedure here.

APPENDIX B: WKB-APPROXIMATION FOR THE GAUGE FIELD IN THE UNIAXIAL CASE

In the WKB-approximation the gauge potentials $A^{(a)}_k$ and the field strengths $F^{(a)}_{kl}$ can be extrapolated as follows

$$A^{(a)}_k \to \mathcal{A}^{(a)}_k e^{i\Psi}, \quad F^{(a)}_{kl} \to i\left[p_k \mathcal{A}^{(a)}_l - p_l \mathcal{A}^{(a)}_k\right] e^{i\Psi}, \quad p_k = \nabla_k \Psi. \tag{B1}$$

Let us mention that the nonlinear terms in (5) give the values of the next order in WKB-approximation, thus, such a model of gauge field is effectively Abelian. In the leading order approximation the Yang-Mills equations are reduced to

$$C^{ikmn}_{(a)(b)}\, p_k\, p_m\, \mathcal{A}^{(b)}_n = 0. \tag{B2}$$

Substitution of $C^{ikmn}_{(a)(b)}$ from (65) with $g^{ik(A)}$ from (78) and $g^{ik(B)}$ from (79) with $\hat{\mu}$ and γ given by (77) yields

$$g^{im(A)}p_m\left[g^{kn(A)}p_k\mathcal{A}^{(a)}_n\right] - g^{in(A)}\mathcal{A}^{(a)}_n\left[g^{km(A)}p_kp_m\right] =$$

$$= \frac{1}{\varepsilon^2_{||}\mu^2_\perp}\left(1 - \frac{\varepsilon_{||}\mu_\perp}{\varepsilon_\perp\mu_{||}}\right)p_kp_m\mathcal{A}^{(a)}_n\left[\left(X^i_{(1)}X^k_{(2)} - X^i_{(2)}X^k_{(1)}\right)\left(X^m_{(1)}X^n_{(2)} - X^m_{(2)}X^n_{(1)}\right)\right]. \tag{B3}$$

Projection of this equations onto the velocity four-vector U_i gives the scalar ratio

$$\left(U^k p_k\right)\left[g^{mn(A)}p_m\mathcal{A}^{(a)}_n\right] = \left(U^n\mathcal{A}^{(a)}_n\right)\left[g^{km(A)}p_kp_m\right], \tag{B4}$$

which is satisfied if, for instance, we use the Landau gauge $U^n\mathcal{A}^{(a)}_n = 0$ and the condition of orthogonality of the wave four-vector and amplitude four-vector in the first associated metric, i.e., $g^{kn}p_k\mathcal{A}^{(a)}_n = 0$. Projections onto the axes, given by $X^i_{(1)}$, $X^i_{(2)}$ and $X^i_{(3)}$ yield, respectively,

$$\mathcal{A}^{(a)}_{(1)}\left[g^{km(A)}p_kp_m + p^2_{(2)}\left(\frac{1}{\varepsilon_{||}\mu_\perp} - \frac{1}{\varepsilon_\perp\mu_{||}}\right)\right] - \mathcal{A}^{(a)}_{(2)}\left[p_{(1)}p_{(2)}\left(\frac{1}{\varepsilon_{||}\mu_\perp} - \frac{1}{\varepsilon_\perp\mu_{||}}\right)\right] = 0,$$

$$\mathcal{A}^{(a)}_{(1)}\left[p_{(1)}p_{(2)}\left(\frac{1}{\varepsilon_{||}\mu_\perp} - \frac{1}{\varepsilon_\perp\mu_{||}}\right)\right] - \mathcal{A}^{(a)}_{(2)}\left[g^{km(A)}p_kp_m + p^2_{(1)}\left(\frac{1}{\varepsilon_{||}\mu_\perp} - \frac{1}{\varepsilon_\perp\mu_{||}}\right)\right] = 0,$$

$$\mathcal{A}^{(a)}_{(3)}\left[g^{km(A)}p_kp_m\right] = 0, \tag{B5}$$

where $\mathcal{A}^{(a)}_{(1)} \equiv X^k_{(1)}\mathcal{A}^{(a)}_k$, $p_{(1)} \equiv X^k_{(1)}p_k$, etc. Taking into account the relation

$$\left[g^{km(B)}p_kp_m\right] = \left[g^{km(A)}p_kp_m\right] + \left(p^2_{(1)} + p^2_{(2)}\right)\left(\frac{1}{\varepsilon_{||}\mu_\perp} - \frac{1}{\varepsilon_\perp\mu_{||}}\right), \tag{B6}$$

one can conclude that nontrivial solution of (B5) exists, when

$$\left[g^{km(A)}p_kp_m\right]\left[g^{lj(B)}p_lp_j\right] = 0. \tag{B7}$$

Thus, the associated metrics $g^{km(A)}$ and $g^{km(B)}$ are the color ones.

APPENDIX C: ALTERNATIVE DESCRIPTION OF THE PARTICLE MOTION IN TERMS OF TIDAL FORCE

Particle dynamics can be alternatively described using the equation of motion with effective force. Instead of equation of null geodesics in the effective spacetime with metric $g^{ik(\alpha)}$, where $\alpha = A, B$, one can write the equation

$$\frac{d^2 x^i}{d\tau^2} + \Gamma^i_{kl} \frac{dx^k}{d\tau} \frac{dx^l}{d\tau} = F^{i(\alpha)} , \qquad (C1)$$

where

$$F^{i(\alpha)} \equiv \Pi^{i(\alpha)}_{kl} \frac{dx^k}{d\tau} \frac{dx^l}{d\tau} , \quad \Pi^{i(\alpha)}_{kl} \equiv \Gamma^i_{kl} - \Gamma^{i(\alpha)}_{kl} . \qquad (C2)$$

Γ^i_{kl} and $\Gamma^{i(\alpha)}_{kl}$ are the Christoffel symbols for the real and effective spacetimes, respectively. The quantity $\Pi^{i(\alpha)}_{kl}$, the difference of the Christoffel symbols symmetric with respect to indices k and l, is known to be a tensor, thus, the quantity $F^{i(\alpha)}$ is a vector. Since we consider the interaction of Yang-Mills and Higgs field with spacetime curvature, we can indicate this force as a tidal one. The tidal force $F^{i(\alpha)}$ is quadratic in the particle velocity four-vector and is predetermined by the structure of the tensor $\Pi^{i(\alpha)}_{kl}$. For Bianchi-I model this tensor has the following non-vanishing components:

$$\Pi^{t(A)}_{11} = \Pi^{t(A)}_{22} = \frac{1}{2} \frac{d}{dt} \left[a^2 \left(1 - \varepsilon_{||} \mu_\perp \right) \right] , \quad \Pi^{t(B)}_{11} = \Pi^{t(B)}_{22} = \frac{1}{2} \frac{d}{dt} \left[a^2 \left(1 - \varepsilon_\perp \mu_{||} \right) \right] ,$$

$$\Pi^{t(A)}_{33} = \Pi^{t(B)}_{33} = \frac{1}{2} \frac{d}{dt} \left[c^2 \left(1 - \varepsilon_\perp \mu_\perp \right) \right] , \quad \Pi^{1(A)}_{1t} = \Pi^{2(A)}_{2t} = -\frac{1}{2} \frac{d}{dt} \ln \left(\varepsilon_{||} \mu_\perp \right) ,$$

$$\Pi^{1(BA)}_{1t} = \Pi^{2(B)}_{2t} = -\frac{1}{2} \frac{d}{dt} \ln \left(\varepsilon_\perp \mu_{||} \right) , \quad \Pi^{3(A)}_{3t} = \Pi^{3(B)}_{3t} = -\frac{1}{2} \frac{d}{dt} \ln \left(\varepsilon_\perp \mu_\perp \right) , \qquad (C3)$$

where the quantities $\varepsilon_{||}$, $\varepsilon_\perp$, $\mu_{||}$ and $\mu_\perp$ are defined in (75) and (76) with the susceptibility tensor components from (89)-(93). Thus, the force $F^{i(\alpha)}$ consists of derivatives of $a(t)$ and $c(t)$ up to the third order.

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